YANG-BAXTER DEFORMATIONS AND RACK COHOMOLOGY

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ABSTRACT. Every rack Q provides a set-theoretic solution c_Q of the Yang-Baxter equation by setting $c_Q\colon x\otimes y\mapsto y\otimes x^y$ for all $x,y\in Q$. This article examines the deformation theory of c_Q within the space of Yang-Baxter operators over a ring $\mathbb{A}$, a problem initiated by Freyd and Yetter in 1989. As our main result we classify deformations in the modular case, which had previously been left in suspense, and establish that every deformation of c_Q is gauge-equivalent to a quasi-diagonal one. Stated informally, in a quasi-diagonal deformation only behaviourally equivalent elements interact. In the extreme case, where all elements of Q are behaviourally distinct, Yang-Baxter cohomology thus collapses to its diagonal part, which we identify with rack cohomology. The latter has been intensively studied in recent years and, in the modular case, is known to produce non-trivial and topologically interesting Yang-Baxter deformations.

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. **Motivation and background.** Yang-Baxter operators (defined in §2) first appeared in theoretical physics, in a 1967 paper by Yang [44] on the many-body problem in one dimension, during the 1970s in work by Baxter [3, 4] on exactly solvable models in statistical mechanics, and later in quantum field theory [19] in connection with the quantum inverse scattering method. They have played a prominent rôle in knot theory and low-dimensional topology ever since the discovery of the Jones polynomial [28] in 1984. Attempts to systematically construct solutions of the Yang-Baxter equation have led to the theory of quantum groups, see Drinfel'd [11] and Turaev, Kassel, Rosso [40, 41, 31, 32].

All Yang-Baxter operators resulting from the quantum approach are deformations of the transposition operator $\tau\colon x\otimes y\mapsto y\otimes x$. As a consequence, the associated knot invariants are of finite type in the sense of Vassiliev [42] and Gusarov [27], see also Birmann–Lin [6] and Bar-Natan [2]. These invariants continue to have a profound impact on low-dimensional topology; their interpretation in terms of algebraic topology and classical knot theory, however, remains difficult and most often mysterious.

As a variation of the theme, Drinfel'd [12] pointed out the interesting special case of *set-theoretic solutions* of the Yang-Baxter equation, see Etingof–Schedler–Soloviev [18] and Lu–Yan–Zhu [36]. One class of solutions is provided by *racks* or *automorphic sets* $(Q,*)$, which have been introduced and thoroughly studied by Brieskorn [7] in the context of braid group actions. Here the operator takes the form $c_Q\colon x\otimes y\mapsto y\otimes x^y$, where $x^y = x*y$ denotes the action of the rack Q on itself. The transposition τ above corresponds to the trivial action; conjugation $x^y = y^{-1}xy$ in a group provides many non-trivial examples.

Applications to knot theory had independently been developed by Joyce [30] and Matveev [38]. Freyd and Yetter [24] observed that the knot invariants obtained from c_Q are the well-known colouring numbers of classical knot theory. These invariants are not of finite type [13]. Freyd and Yetter [24, 45] also initiated the natural question of deforming set-theoretic solutions within the space of Yang-Baxter operators over a ring $\mathbb{A}$, and illustrated their general approach by the simplified ansatz of diagonal deformations [24, §4]. The latter are encoded by rack cohomology, which was independently developed by Fenn and Rourke [21] from a homotopy-theoretic viewpoint via classifying spaces.

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Carter et al. [9] have applied rack and quandle cohomology to knots by constructing state-sum invariants. These, in turn, can be interpreted in terms of classical algebraic topology as *colouring polynomials* associated to knot group representations [16].

For more recent developments and open questions see §8 at the end of this article.

1.2. **Yang-Baxter deformations.** In the present article we continue the study of Yang-Baxter deformations of racks linearized over a ring $\mathbb{A}$, as initiated by Freyd and Yetter [24]. Detailed definitions will be given in §2 below, in particular we will review Yang-Baxter operators (§2.1) and set-theoretic solutions c_Q coming from racks (§2.2). In this introduction we merely recall the basic definitions in order to state our main result.

Notation (rings and modules)**.** Throughout this article $\mathbb{A}$ denotes a commutative ring with unit. All modules will be $\mathbb{A}$-modules, all maps between modules will be $\mathbb{A}$-linear, and all tensor products will be formed over $\mathbb{A}$. For every $\mathbb{A}$-module V we denote by $V^{\otimes n}$ the tensor product $V \otimes \cdots \otimes V$ of n copies of V. Given a set Q we denote by $\mathbb{A}Q$ the free $\mathbb{A}$-module with basis Q. We identify the n-fold tensor product $(\mathbb{A}Q)^{\otimes n}$ with $\mathbb{A}Q^n$. In particular, this choice of bases allows us to identify $\mathbb{A}$-linear maps $\mathbb{A}Q^n \to \mathbb{A}Q^n$ with matrices $Q^n \times Q^n \to \mathbb{A}$.

For the purposes of deformation theory we equip $\mathbb{A}$ with a fixed ideal $\mathfrak{m} \subset \mathbb{A}$. Most often we require that $\mathbb{A}$ be complete with respect to $\mathfrak{m}$, that is, we assume that the natural map $\mathbb{A} \to \varprojlim \mathbb{A}/\mathfrak{m}^n$ is an isomorphism. A typical setting is the power series ring $\mathbb{K}[\![h]\!]$ over a field $\mathbb{K}$, equipped with its maximal ideal $\mathfrak{m} = (h)$, or the ring of p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/_{p^n}$ with its maximal ideal (p).

Notation (racks)**.** A *rack* or *automorphic set* $(Q,*)$ is a set Q equipped with an operation $*\colon Q \times Q \to Q$ such that every right translation $x \mapsto x * y$ is an automorphism of $(Q,*)$. This is equivalent to saying that the $\mathbb{A}$-linear map $c_Q\colon \mathbb{A}Q \otimes \mathbb{A}Q \to \mathbb{A}Q \otimes \mathbb{A}Q$ defined by $c_Q\colon x \otimes y \mapsto y \otimes (x * y)$ for all $x, y \in Q$ is a Yang-Baxter operator over the ring $\mathbb{A}$.

Two rack elements $y, z \in Q$ are called *behaviourally equivalent*, denoted $y \equiv z$, if they satisfy $x * y = x * z$ for all $x \in Q$. This is equivalent to saying that y, z have the same image under the inner representation $\rho\colon Q \to \mathrm{Inn}(Q)$. As usual, a matrix $f\colon Q^n \times Q^n \to \mathbb{A}$ is called *diagonal* if $f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right]$ vanishes whenever $x_i \neq y_i$ for some index $i = 1, \dots, n$. It is called *quasi-diagonal* if $f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right]$ vanishes whenever $x_i \not\equiv y_i$ for some index $i = 1, \dots, n$.

Quasi-diagonal maps play a crucial rôle in the classification of deformations:

Theorem 1.1. *If the ring $\mathbb{A}$ is complete with respect to the ideal $\mathfrak{m}$, then every Yang-Baxter deformation c of c_Q over $(\mathbb{A}, \mathfrak{m})$ is equivalent to a quasi-diagonal deformation. More explicitly this means that c is conjugated over $(\mathbb{A}, \mathfrak{m})$ to a deformation of the form $c_Q \circ (\mathrm{id}^{\otimes 2} + f)$ where the deformation term $f\colon \mathbb{A}Q^2 \to \mathfrak{m}Q^2$ is quasi-diagonal.*

There are thus two extreme cases in the deformation theory of racks:

(1) In the one extreme the rack Q is trivial, whence $\rho\colon Q \to \mathrm{Inn}(Q)$ is trivial and all elements of Q are behaviourally equivalent. This is the initial setting in the theory of quantum invariants and we cannot add anything new here.
(2) In the other extreme, where $\rho\colon Q \to \mathrm{Inn}(Q)$ is injective, all elements of Q are behaviourally distinct, and every deformation of c_Q is equivalent to a diagonal deformation. This is the setting of rack cohomology and colouring invariants.

This result confirms a plausible observation: the more innner symmetries Q has, the less deformations c_Q admits. Our theorem makes the transition between the two extremes precise and quantifies the degree of deformability of set-theoretic Yang-Baxter operators.

Example 1.2. Consider a group $(G, \cdot)$ that is generated by one of its conjugacy classes $Q \subset G$. Then $(Q,*)$ is a rack with respect to conjugation $x * y = y^{-1} \cdot x \cdot y$, and we have a natural isomorphism $\mathrm{Inn}(Q,*) \cong \mathrm{Inn}(G,\cdot)$. If the centre of G is trivial, then $\rho\colon G \xrightarrow{\sim} \mathrm{Inn}(G)$ is a group isomorphism. For the operator c_Q the injectivity of $\rho\colon Q \to \mathrm{Inn}(Q)$ implies that

every Yang-Baxter deformation of c_Q is equivalent to a diagonal deformation. If, moreover, the order $|G|$ is finite and invertible in $\mathbb{A}$, then c_Q is rigid over $\mathbb{A}$.

As pointed out above, diagonal deformations have received much attention over the last 20 years [24, 21, 45, 9]. It is reassuring that Theorem 1.1 justifies this short-cut in the case where $\rho\colon Q \to \operatorname{Inn}(Q)$ is injective. In general, however, the simplified ansatz of diagonal deformations may miss some interesting Yang-Baxter deformations, namely those that are quasi-diagonal but not diagonal. For more detailed examples and applications see §7.

1.3. **Yang-Baxter cohomology.** Our approach to proving Theorem 1.1 follows the classical paradigm of studying algebraic deformation theory via cohomology, as expounded by Gerstenhaber [25]. Since it may be of independent interest, we state here our main cohomological result, which in degree 2 proves the infinitesimal version of Theorem 1.1.

In the previous article [15] I introduced Yang-Baxter cohomology $H^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$, which encodes infinitesimal deformations of c_Q over a ring $\mathbb{A}$ with respect to the ideal $\mathfrak{m}$ (§2.3). There I calculated the second cohomology $H^2_{\mathrm{YB}}(c_Q;\mathfrak{m})$ under the hypothesis that the order of inner automorphism group $\operatorname{Inn}(Q)$ is finite and invertible in the ring $\mathbb{A}$. The main application was to finite racks Q deformed over the ring $\mathbb{A} = \mathbb{Q}[\![h]\!]$. In many cases the results of [15] imply that c_Q is rigid over $\mathbb{Q}[\![h]\!]$.

In the present article we calculate Yang-Baxter cohomology $H^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$ in the modular case, which had previously been left in suspense [15, Question 39]. As our main result we establish the following classification; for detailed definitions and proofs we refer to §5.

Theorem 1.3. *The quasi-diagonal subcomplex $C^*_\Delta(c_Q;\mathfrak{m}) \subset C^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$ is a homotopy retract, whence the induced map $H^*_\Delta(c_Q;\mathfrak{m}) \to H^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$ is an isomorphism.*

Notice that, contrary to [15], we no longer require the rack Q to be finite, nor do we impose any restrictions on the characteristic of the ring $\mathbb{A}$.

Remark 1.4. Yang-Baxter cohomology includes rack cohomology $H^*_{\mathrm{R}}(Q;\Lambda)$ as its diagonal part, as explained in §3, where Λ is a module over some ring $\mathbb{K}$. If $|\operatorname{Inn}(Q)|$ is invertible in $\mathbb{K}$, then $H^*_{\mathrm{R}}(Q;\Lambda)$ is trivial in a certain sense [17]. In the modular case, however, it leads to non-trivial and topologically interesting Yang-Baxter deformations (see Example 7.7 below). It follows that Yang-Baxter cohomology of racks must be at least as complicated, and the modular case stood out as a difficult yet promising problem.

Theorem 1.3 solves this problem: it shows that the right object to study is the quasi-diagonal subcomplex C^*_Δ, situated between the strictly diagonal complex C^*_{Diag} and the full Yang-Baxter complex C^*_{YB}, i.e., we have $C^*_{\mathrm{Diag}} \subset C^*_\Delta \subset C^*_{\mathrm{YB}}$. We will see that the inclusion $C^*_{\mathrm{Diag}} \subset C^*_{\mathrm{YB}}$ allows for a retraction $C^*_{\mathrm{YB}} \twoheadrightarrow C^*_{\mathrm{Diag}}$, which entails that H^*_{Diag} is a direct summand of H^*_{YB}. In general, however, this is not a homotopy retraction and $H^*_{\mathrm{Diag}} \subsetneqq H^*_{\mathrm{YB}}$. The inclusion $\iota\colon C^*_\Delta \subset C^*_{\mathrm{YB}}$ also allows for a retraction $\pi\colon C^*_{\mathrm{YB}} \twoheadrightarrow C^*_\Delta$, such that $\pi \circ \iota = \mathrm{id}_\Delta$, and our main result is the construction of a homotopy $\iota \circ \pi \simeq \mathrm{id}_{\mathrm{YB}}$.

Remark 1.5. Again we have two extreme cases that are particularly clear-cut:

(1) In the one extreme, if Q is trivial, then all elements of Q are behaviourally equivalent. In this case we trivially have $C_\Delta = C_{\mathrm{YB}}$.
(2) If $\rho\colon Q \to \operatorname{Inn}(Q)$ is injective, then all elements of Q are behaviourally distinct. In this case quasi-diagonal means diagonal, whence $C_\Delta = C_{\mathrm{Diag}}$.

In general C^*_Δ lies strictly between C^*_{Diag} and C^*_{YB}. Even if C_Δ collapses to C_{Diag}, this restrictive situation is in general not rigid, and interesting deformations do arise in the modular case (see Example 7.7).

Depending on the structure of the rack Q, the quasi-diagonal subcomplex $C^*_\Delta(c_Q;\mathfrak{m})$ can still be quite big, but in any case collapsing the full Yang-Baxter complex $C^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$ to its quasi-diagonal subcomplex greatly simplifies the problem. In practical terms it reduces the complexity from $|Q|^4$ unknowns to the order of $|Q|^2$ unknowns, which in many cases makes it amenable to computer calculations. See §7 for illustrating examples.

1.4. **How this article is organized.** Section 2 recollects the relevant definitions concerning Yang-Baxter operators, their deformations, and the corresponding cohomology theory. It also gives explicit formulae in the case of racks, which is our main focus here. Section 3 identifies diagonal deformation with rack cohomology. Section 4 clarifies the relationship between diagonal deformations and rack cohomology, and introduces quasi-diagonal deformations. Section 5 proves our main result in the infinitesimal case, by constructing a homotopy retraction from the Yang-Baxter complex to its quasi-diagonal subcomplex. Section 6 extends the infinitesimal result to complete deformations, and Section 7 provides some examples and applications. We conclude with some open question in Section 8.

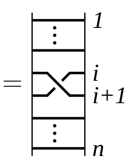
2. Yang-Baxter operators, deformations, and cohomology

2.1. Yang-Baxter operators.

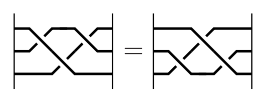
Definition 2.1. A *Yang-Baxter operator* is an automorphism $c\colon V \otimes V \to V \otimes V$ satisfying the *Yang-Baxter equation*, also called *braid relation*,

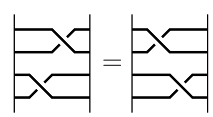
$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c) \qquad \text{in} \quad \mathrm{Aut}_{\mathbb{A}}(V^{\otimes 3}). \tag{2.1}$$

This equation first appeared in theoretical physics (Yang [44], Baxter [3, 4], Faddeev [19]). It also has a very natural interpretation in terms of Artin's braid group B_n [1, 5] and its tensor representations: the automorphisms $c_1, \dots, c_{n-1}\colon V^{\otimes n} \to V^{\otimes n}$ defined by

$$c_i = \mathrm{id}_V^{\otimes(i-1)} \otimes c \otimes \mathrm{id}_V^{\otimes(n-i-1)}, \qquad \text{or in graphical notation } c_i =$$  ,

satisfy the well-known braid relations

$$c_i c_j c_i = c_j c_i c_j \quad \text{if } |i-j| = 1,$$ 

$$c_i c_j = c_j c_i \quad \text{if } |i-j| \geq 2,$$ 

Remark 2.2. A graphical notation for tensor calculus was first used by Penrose [39]; for a brief discussion of its history see [29]. This notation has the obvious advantage to appeal to our geometric vision. More importantly, it incorporates a profound relationship with knot theory, and its rigorous formulation in terms of tensor categories directly leads to knot invariants. More explicitly, Yang-Baxter operators induce invariants of knots and links in $\mathbb{S}^3$ as follows, see Turaev [41, chap. I] or Kassel [31, chap. X].

$$\begin{tikzcd} B_n \arrow[r, "\text{Yang-Baxter}", "\text{representation}"'] \arrow[d, "\text{closure}"'] & \mathrm{Aut}(V^{\otimes n}) \arrow[d, "\text{trace}"] \\ \{\text{links}\} \arrow[r, "\text{invariant}"'] & \mathbb{A} \end{tikzcd}$$

Each link L can be presented as the closure of some braid. This braid acts on $V^{\otimes n}$ as defined above, and a suitably deformed trace maps it to the ring $\mathbb{A}$. In favourable cases the result does not depend on the choice of braid, and thus defines an invariant of the link L.

2.2. **Quandles and racks.** In every group $(G,\cdot)$ the conjugation $a*b = b^{-1}\cdot a\cdot b$ enjoys the following properties:

(Q1) For every element a we have $a*a=a$. (idempotency)
(Q2) Every right translation $\rho(b)\colon a\mapsto a*b$ is a bijection. (right invertibility)
(Q3) For all a,b,c we have $(a*b)*c=(a*c)*(b*c)$. (self-distributivity)

Taking these properties as axioms, Joyce [30] defined a *quandle* to be a set Q equipped with a binary operation $*\colon Q\times Q\to Q$ satisfying (Q1–Q3). Independently, Matveev [38] studied the equivalent notion of *distributive groupoid*. Following Brieskorn [7], an *automorphic set* is only required to satisfy (Q2–Q3): these two axioms are equivalent to saying that every right translation is an automorphism of $(Q,*)$. The shorter term *rack* was suggested by Fenn and Rourke [21], going back to the terminology used by J.H. Conway in correspondence with G.C. Wraith in 1959. Such structures appear naturally in the study of braid actions [7] and provide set-theoretic solutions of the Yang-Baxter equation [12]:

Proposition 2.3. *Given a set Q with binary operation $*\colon Q\times Q\to Q$, we can consider the free module $V=\mathbb{A}Q$ with basis Q over $\mathbb{A}$ and define the operator*

$$c_Q\colon \mathbb{A}Q\otimes\mathbb{A}Q\to\mathbb{A}Q\otimes\mathbb{A}Q \quad\text{by}\quad a\otimes b\mapsto b\otimes(a*b) \quad\text{for all}\quad a,b\in Q.$$

Then c_Q is a Yang-Baxter operator if and only if Q is a rack. □

Throughout this article we will use the following notation. A *homomorphism* between two racks $(Q,*)$ and $(Q',*')$ is a map $\phi\colon Q\to Q'$ satisfying $\phi(a*b)=\phi(a)*'\phi(b)$ for all $a,b\in Q$. Racks and their homomorphisms form a category.

The *automorphism group* $\operatorname{Aut}(Q)$ consists of all bijective homomorphisms $\phi\colon Q\to Q$. We adopt the convention that automorphisms of Q act on the right, written a^ϕ, which means that their composition $\phi\psi$ is defined by $a^{(\phi\psi)}=(a^\phi)^\psi$ for all $a\in Q$.

Each $a\in Q$ defines an automorphism $\rho(a)\in\operatorname{Aut}(Q)$ defined by $x\mapsto x*a$. For every $\phi\in\operatorname{Aut}(Q)$ we have $\rho(a^\phi)=\rho(a)^\phi$. The group $\operatorname{Inn}(Q)$ of *inner automorphisms* is the normal subgroup of $\operatorname{Aut}(Q)$ generated by all right translations $\rho(a)$, where $a\in Q$. The map $\rho\colon Q\to\operatorname{Inn}(Q)$ is the *inner representation* of Q. It satisfies $\rho(a*b)=\rho(a)*\rho(b)$, that is, it maps the operation of the rack Q to conjugation in the group $\operatorname{Inn}(Q)$.

Notation. In view of the representation $\rho\colon Q\to\operatorname{Inn}(Q)$, we often write a^b for the operation $a^{\rho(b)}=a*b$. Conversely, it will sometimes be convenient to write $a*b$ for the conjugation $a^b=b^{-1}ab$ in a group. In neither case will there be any danger of confusion.

Definition 2.4. Two elements $x,y\in Q$ are *behaviourally equivalent* if $a*x=a*y$ for all $a\in Q$. This means that $\rho(x)=\rho(y)$, and will be denoted by $x\equiv y$ for short.

2.3. **Deformations and Yang-Baxter cohomology.** We are interested here in set-theoretic solutions of the Yang-Baxter equation and their deformations within the space of Yang-Baxter operators over some ring. A typical setting is the power series ring $\mathbb{K}[\![h]\!]$ over a field $\mathbb{K}$, equipped with its maximal ideal $\mathfrak{m}=(h)$. In positive characteristic it is equally natural to consider the ring of p-adic integers $\mathbb{Z}_p=\varprojlim\mathbb{Z}/_{p^n}$ with its maximal ideal (p).

Definition 2.5. We fix an ideal $\mathfrak{m}$ in the ring $\mathbb{A}$. Consider an $\mathbb{A}$-module V and a Yang-Baxter operator $c\colon V\otimes V\to V\otimes V$.

A map $\tilde c\colon V\otimes V\to V\otimes V$ is called a *Yang-Baxter deformation* of c with respect to $\mathfrak{m}$ if $\tilde c$ is itself a Yang-Baxter operator and satisfies $\tilde c\equiv c$ modulo $\mathfrak{m}$.

An *equivalence transformation*, or *gauge equivalence* with respect to $\mathfrak{m}$, is an automorphism $\alpha\colon V\to V$ satisfying $\alpha\equiv\mathrm{id}_V$ modulo $\mathfrak{m}$.

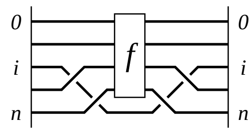
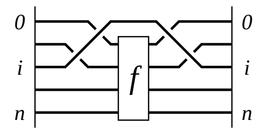
Two Yang-Baxter operators c and $\tilde c$ are called *equivalent* if there exists an equivalence transformation $\alpha\colon V\to V$ such that $\tilde c=(\alpha\otimes\alpha)^{-1}\circ c\circ(\alpha\otimes\alpha)$.

In order to study deformations it is useful, as usual, to linearize the problem by considering infinitesimal deformations, where $\mathfrak{m}^2=0$. To this end we recall the definition of Yang-Baxter cohomology $H_{\mathrm{YB}}(c;\mathfrak{m})$ that encodes infinitesimal deformations.

Definition 2.6. The Yang-Baxter cochain complex $C^*_{\mathrm{YB}}(c;\mathfrak{m})$ consists of the $\mathbb{A}$-modules $C^n = \mathrm{Hom}(V^{\otimes n}, \mathfrak{m}V^{\otimes n})$. For each $f \in C^n$ we define the partial coboundary $d_i^n f \in C^{n+1}$ by

$$d_i^n f = (c_n \cdots c_{i+1})^{-1}\,(f \otimes \mathrm{id}_V)\,(c_n \cdots c_{i+1}) - (c_1 \cdots c_i)^{-1}\,(\mathrm{id}_V \otimes f)\,(c_1 \cdots c_i)\,. \tag{2.2}$$

This formula becomes more suggestive in graphical notation:

(2.3) $\qquad d_i^n f = \quad +$  $-$  .

The coboundary $d^n \colon C^n \to C^{n+1}$ is defined as the alternating sum $d^n = \sum_{i=0}^n (-1)^i d_i^n$.

Proposition 2.7. *We have $d_i^{n+1} \circ d_j^n = d_{j+1}^{n+1} \circ d_i^n$ for $i \le j$, whence $d^{n+1} \circ d^n = 0$.*

Proof. The hypothesis that c be a Yang-Baxter operator ensures that $d_i^{n+1} d_j^n = d_{j+1}^{n+1} d_i^n$ for $i \le j$. This can be proven by a straightforward computation, and is most easily verified using the graphical calculus suggested above. It follows, as usual, that all terms cancel each other in pairs to yield $d^{n+1} \circ d^n = 0$. $\square$

Definition 2.8. The cochain complex $C^*_{\mathrm{YB}}(c;\mathfrak{m}) = (C^*, d^*)$ is called the *Yang-Baxter cochain complex*, and its cohomology $H^*_{\mathrm{YB}}(c;\mathfrak{m})$ is called the *Yang-Baxter cohomology* of the operator c with respect to the ideal $\mathfrak{m}$.

Proposition 2.9. *The second cohomology $H^2_{\mathrm{YB}}(c;\mathfrak{m})$ classifies infinitesimal Yang-Baxter deformations: assuming $\mathfrak{m}^2 = 0$, the deformation $\tilde{c} = c \circ (\mathrm{id}_V^{\otimes 2} + f)$ satisfies*

$$\begin{aligned}
&(\mathrm{id}_V \otimes \tilde{c})^{-1}(\tilde{c} \otimes \mathrm{id}_V)^{-1}(\mathrm{id}_V \otimes \tilde{c})^{-1}(\tilde{c} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \tilde{c})(\tilde{c} \otimes \mathrm{id}_V) \\
&\qquad = (\mathrm{id}_V \otimes c)^{-1}(c \otimes \mathrm{id}_V)^{-1}(\mathrm{id}_V \otimes c)^{-1}(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) + d^2 f.
\end{aligned}$$

This means that $\tilde{c}$ is a Yang-Baxter operator if and only if $d^2 f = 0$. Likewise, c and $\tilde{c}$ are equivalent via conjugation by $\alpha = (\mathrm{id}_V + g)$ if and only if $f = d^1 g$, because

$$(\alpha \otimes \alpha)^{-1} \circ c \circ (\alpha \otimes \alpha) = c \circ (\mathrm{id}_V^{\otimes 2} + d^1 g).$$

Remark 2.10. In the quantum case, where $c = \tau$, we obtain $df = 0$ for all $f \in C^*_{\mathrm{YB}}$. In particular there are no infinitesimal obstructions to deforming τ: *every* deformation of τ satisfies the Yang-Baxter equation modulo $\mathfrak{m}^2$, and only higher-order obstructions are of interest. This explains why Yang-Baxter cohomology is absent in the quantum case.

Infinitesimal obstructions are important, however, if $c \ne \tau$, for example for an operator c_Q coming from a non-trivial rack Q, the main object of interest to us here. In extreme cases they even allow us to conclude that c_Q is rigid.

Example 2.11. Yang-Baxter cohomology can in particular be applied to study the deformations of the Yang-Baxter operator c_Q associated with a rack Q. The canonical basis Q of $V = \mathbb{A}Q$ allows us to identify each $\mathbb{A}$-linear map $f \colon \mathbb{A}Q^n \to \mathbb{A}Q^n$ with its matrix $f \colon Q^n \times Q^n \to \mathbb{A}$, related by the definition

$$f \colon\; (x_1 \otimes \cdots \otimes x_n) \mapsto \sum_{y_1,\dots,y_n} f\begin{bmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{bmatrix} \cdot (y_1 \otimes \cdots \otimes y_n)\,.$$

Matrix entries are thus denoted by $f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right]$ with indices $\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right] \in Q^n \times Q^n$. If Q is infinite, then we use the tacit convention that for each basis element $(x_1,\dots,x_n) \in Q^n$ the coefficient $f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right]$ is non-zero only for a finite number of $(y_1,\dots,y_n) \in Q^n$.

For example, the identity $\mathrm{id} \colon \mathbb{A}Q \to \mathbb{A}Q$ will be identified with the following matrix $Q \times Q \to \mathbb{A}$, which is usually called the Kronecker delta function:

$$\mathrm{id}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \ne y. \end{cases}$$

In this notation the coboundary can be rewritten more explicitly as follows:

$$(2.4)\qquad (d_i^n f)\begin{bmatrix} x_0,\dots,x_n \\ y_0,\dots,y_n \end{bmatrix} = +f\begin{bmatrix} x_0\,,\dots,x_{i-1},x_{i+1},\dots,x_n \\ y_0\,,\dots,y_{i-1},y_{i+1},\dots,y_n \end{bmatrix}\cdot \mathrm{id}\begin{bmatrix} x_i^{x_{i+1}\cdots x_n} \\ y_i^{y_{i+1}\cdots y_n} \end{bmatrix}$$
$$-f\begin{bmatrix} x_0^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,x_n \\ y_0^{y_i},\dots,y_{i-1}^{y_i},y_{i+1},\dots,y_n \end{bmatrix}\cdot \mathrm{id}\begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

Remark 2.12. Instead of an ideal $\mathfrak{m}$ in a ring $\mathbb{A}$ one can also define the Yang-Baxter cochain complex $C^*_{\mathrm{YB}}(c;\mathfrak{m})$ and its cohomology $H^*_{\mathrm{YB}}(c;\mathfrak{m})$ for any module $\mathfrak{m}$ over a ring $\mathbb{K}$. Both points of view become equivalent in the infinitesimal setting:

- First, if $\mathfrak{m}^2 = 0$ in $\mathbb{A}$, then $\mathfrak{m}$ is a module over the quotient ring $\mathbb{K} = \mathbb{A}/\mathfrak{m}$.
- Conversely, every $\mathbb{K}$-module $\mathfrak{m}$ defines a $\mathbb{K}$-algebra $\mathbb{A} = \mathbb{K}\oplus\mathfrak{m}$ with $\mathfrak{m}^2 = 0$.

Consider for example the power series ring $\mathbb{K}[\![h]\!]$ over a ring $\mathbb{K}$. In the infinitesimal setting we have $\mathbb{A} = \mathbb{K}[\![h]\!]/(h^2) = \mathbb{K}\oplus\mathfrak{m}$ where $\mathfrak{m} = \mathbb{K}h$ is isomorphic with $\mathbb{K}$.

The p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/_{p^n}$ lead to the infinitesimal algebra $\mathbb{A} = \mathbb{Z}_p/(p^2) = \mathbb{Z}/_{p^2}$. Here $\mathfrak{m} = (p) \cong \mathbb{Z}/_p$ and $\mathbb{K} = \mathbb{A}/\mathfrak{m} = \mathbb{Z}/_p$, but the projection $\mathbb{A} \twoheadrightarrow \mathbb{K}$ does not split.

This shows that an infinitesimal algebra $\mathbb{A}$, with $\mathfrak{m}^2 = 0$, need not be of the form $\mathbb{K}\oplus\mathfrak{m}$. This interesting phenomenon only appears in positive characteristic. It does not play any rôle for infinitesimal deformations, but may become crucial for higher order deformations.

2.4. **Yang-Baxter homology.** As could be expected, there is a homology theory dual to Yang-Baxter cohomology. Even though we shall not explicitly use it in the sequel, it may be illuminating to briefly sketch its construction. Since we are interested in analyzing coefficient modules, the standard approach would be to exploit the interplay between homology and cohomology via the Universal Coefficient Theorem, see [37], §III.4 and §V.11.

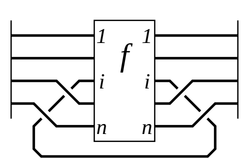
Remark 2.13 (traces)**.** Let $\mathbb{A}$ be a ring and let $c\colon V\otimes V \to V\otimes V$ be a Yang-Baxter operator. We will assume that the $\mathbb{A}$-module V is free of finite rank, so that we can define a trace $\mathrm{tr}\colon \mathrm{End}(V) \to \mathbb{A}$, see Lang [35, §XVI.5]. Slightly more general, it suffices to assume V projective and finitely generated over $\mathbb{A}$, see Turaev [41, chap. 1]. Even though this hypothesis may seem restrictive, it is precisely the setting of quantum knot invariants, where a trace $\mathrm{tr}\colon \mathrm{End}(V) \to \mathbb{A}$ is indispensable. Notice further that then $\mathrm{End}(V^{\otimes n}) = \mathrm{End}(V)^{\otimes n}$, and for each index $i = 1,\dots,n$ we have a partial trace $\mathrm{tr}_i\colon \mathrm{End}(V)^{\otimes n} \to \mathrm{End}(V)^{\otimes(n-1)}$ defined by contracting the ith tensor factor.

Definition 2.14. Given a Yang-Baxter operator $c\colon V\otimes V \to V\otimes V$, the Yang-Baxter chain complex $C_*^{\mathrm{YB}}(c)$ consists of the $\mathbb{A}$-modules $C_n = \mathrm{End}(V^{\otimes n})$. We define the partial boundary $\partial_n^i\colon C_n \to C_{n-1}$ by

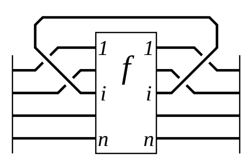
$$(2.5)\quad \partial_n^i f = \mathrm{tr}_n\left[(c_{n-1}\cdots c_i)\circ f\circ (c_{n-1}\cdots c_i)^{-1}\right] - \mathrm{tr}_1\left[(c_1\cdots c_{i-1})\circ f\circ (c_1\cdots c_{i-1})^{-1}\right].$$

Again this formula becomes more suggestive in graphical notation, which is, as (2.3), reminiscent of rope skipping:

$$(2.6)\qquad \partial_n^i f = \quad +$$



$$-$$



.

The boundary $\partial_n\colon C_n \to C_{n-1}$ is defined as the alternating sum $\partial_n = \sum_{i=1}^n (-1)^{i-1}\partial_n^i$.

Proposition 2.15. *We have $\partial_{n-1}^j \circ \partial_n^i = \partial_{n-1}^i \circ \partial_n^{j+1}$ for $i \le j$, whence $\partial_{n-1}\circ\partial_n = 0$.*

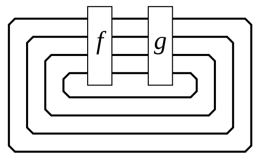
Proof. The hypothesis that c be a Yang-Baxter operator ensures that $\partial_{n-1}^j \circ \partial_n^i = \partial_{n-1}^i \circ \partial_n^{j+1}$ for $i \le j$. This can be proven by a straightforward computation, and is most easily verified

using the graphical calculus suggested above. It follows, as usual, that all terms cancel each other in pairs to yield $\partial_{n-1} \circ \partial_n = 0$. □

Definition 2.16. The chain complex $C^{\mathrm{YB}}_*(c) = (C_*, \partial_*)$ is called the *Yang-Baxter chain complex*, and its homology $H^{\mathrm{YB}}_*(c)$ is called the *Yang-Baxter homology* of the operator c.

Proposition 2.17. *The dual complex* $\mathrm{Hom}(C^{\mathrm{YB}}_*, \mathfrak{m})$ *is naturally isomorphic to the Yang-Baxter cochain complex* $C^*_{\mathrm{YB}}(c;\mathfrak{m})$ *defined above.*

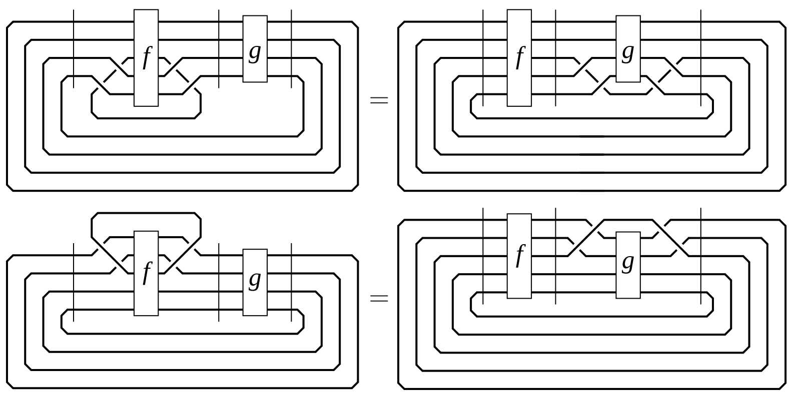
Proof. The natural duality is induced by the duality pairing $\mathrm{End}(V^{\otimes n}) \otimes \mathrm{End}(V^{\otimes n}) \to \mathbb{A}$ defined by $\langle f \mid g \rangle = \mathrm{tr}(fg)$. In graphical notation this reads as

$\langle f \mid g \rangle =$  .

The advantage of this notation is that all calculations become self-evident. In particular, we see that the coboundary operator d^* of Equation (2.3) is the dual of the boundary operator ∂_* of Equation (2.6): for $f \in C^{\mathrm{YB}}_{n+1}$ and $g \in C^n_{\mathrm{YB}}$ and all $i = 1, \dots, n+1$ we have

$$\langle \partial^i_{n+1} f \mid g \rangle = \langle f \mid d^n_{i-1} g \rangle.$$

In graphical notation this can be seen as follows:



We conclude that $\langle \partial_{n+1} f \mid g \rangle = \langle f \mid d^n g \rangle$ as claimed. □

Remark 2.18. In the case of a finite rack Q and its associated Yang-Baxter operator c_Q, the chain complex C^{YB}_* can be described as follows. Starting from the canonical basis Q of $V = \mathbb{A}Q$, we obtain the basis Q^n of $V^{\otimes n}$ and then a basis $Q^n \times Q^n$ of $\mathrm{End}(V^{\otimes n})$. In analogy with our previous notation we denote by $\binom{x_1,\dots,x_n}{y_1,\dots,y_n}$ the endomorphism that maps $x_1 \otimes \cdots \otimes x_n$ to $y_1 \otimes \cdots \otimes y_n$, while mapping all other elements of the basis Q^n to zero. The boundary operator can then be rewritten more explicitly as follows:

$$\partial_n \begin{pmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{pmatrix} = \sum_{i=1}^{n} (-1)^{i-1} \Bigg[\begin{pmatrix} x_1,\dots,x_{i-1},x_{i+1},\dots,x_n \\ y_1,\dots,y_{i-1},y_{i+1},\dots,y_n \end{pmatrix} \cdot \mathrm{id} \begin{pmatrix} x_i^{x_{i+1}\cdots x_n} \\ y_i^{y_{i+1}\cdots y_n} \end{pmatrix} - \begin{pmatrix} x_1^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,x_n \\ y_1^{y_i},\dots,y_{i-1}^{y_i},y_{i+1},\dots,y_n \end{pmatrix} \cdot \mathrm{id} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \Bigg]. \tag{2.7}$$

We see that the boundary formula (2.7) is dual to the coboundary formula (2.4), which nicely illustrates the preceding proposition. Again diagonal chains form a subcomplex, which corresponds to rack homology defined in [21, 10]. The diagonal subcomplex $C^{\mathrm{Diag}}_* \subset C^{\mathrm{YB}}_*$ is a retract, so that rack homology is a direct summand of Yang-Baxter homology. Moreover, under the above duality, $C^{\mathrm{Diag}}_* \subset C^{\mathrm{YB}}_*$ is dual to $C^*_{\mathrm{Diag}} \subset C^*_{\mathrm{YB}}$.

Remark 2.19. The duality exhibited above is graphically appealing and theoretically satisfying: it is reassuring to have the standard homology-cohomology pairing. Notice, however, that we have to restrict to free modules of finite rank, or finitely generated projective modules. Yang-Baxter cohomology alone can be defined over arbitrary Yang-Baxter modules (V, c), not necessarily projective or finitely generated. From this abstract viewpoint Yang-Baxter cohomology thus seems more natural than homology.

2.5. **Non-Functoriality.** Yang-Baxter cohomology and homology suffer from a curious defect: they are not functorial with respect to homomorphisms of Yang-Baxter operators:

Definition 2.20. A *homomorphism* between Yang-Baxter operators $c\colon V\otimes V\to V\otimes V$ and $\bar{c}\colon \bar{V}\otimes\bar{V}\to\bar{V}\otimes\bar{V}$ is an $\mathbb{A}$-linear map $\phi\colon V\to\bar{V}$ such that $\bar{c}\circ(\phi\otimes\phi)=(\phi\otimes\phi)\circ c$.

$$\begin{tikzcd} V\otimes V \arrow[r, "c"] \arrow[d, "\phi\otimes\phi"'] & V\otimes V \arrow[d, "\phi\otimes\phi"] \\ \bar{V}\otimes\bar{V} \arrow[r, "\bar{c}"] & \bar{V}\otimes\bar{V} \end{tikzcd}$$

This condition ensures that ϕ induces for each n a homomorphism $\phi^{\otimes n}\colon V^{\otimes n}\to\bar{V}^{\otimes n}$ that is equivariant with respect to the natural action of Artin's braid group B_n.

Example 2.21. A map $\phi\colon Q\to\bar{Q}$ is a homomorphism between two racks Q and $\bar{Q}$ if and only if only if its $\mathbb{A}$-linear extension $\phi\colon\mathbb{A}Q\to\mathbb{A}\bar{Q}$ is a homomorphism between the associated Yang-Baxter operators c_Q and $c_{\bar{Q}}$.

Remark 2.22. Given a homomorphism ϕ between Yang-Baxter operators c and $\bar{c}$, we would expect a natural cochain homomorphism $\phi^*\colon C^*_{\mathrm{YB}}(\bar{c};\mathfrak{m})\to C^*_{\mathrm{YB}}(c;\mathfrak{m})$ as well as a natural chain homomorphism $\phi_*\colon C_*^{\mathrm{YB}}(c)\to C_*^{\mathrm{YB}}(\bar{c})$. The definitions of $C_n^{\mathrm{YB}}(c)=\mathrm{End}(V^{\otimes n})$ and $C^n_{\mathrm{YB}}(c;\mathfrak{m})=\mathrm{Hom}(V^{\otimes n},\mathfrak{m}V^{\otimes n})$, however, do not lend themselves to any obvious construction. This difficulty persists even if V is free of finite rank. In degree 2 the problem is that in a general deformation of $c\colon V\otimes V\to V\otimes V$ both factors interact non-trivially. This does not respect the product structure of $\phi^{\otimes n}$.

To be more explicit, consider a homomorphism $\phi\colon Q\to\bar{Q}$ between finite racks. We can define a map $\phi^*\colon C^*_{\mathrm{YB}}(c_{\bar{Q}};\mathfrak{m})\to C^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$ by setting

$$(\phi^* f)\begin{bmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{bmatrix} = f\begin{bmatrix} \phi(x_1),\dots,\phi(x_n) \\ \phi(y_1),\dots,\phi(y_n) \end{bmatrix}. \tag{2.8}$$

Even though this is the natural candidate, it does in general not define a cochain map, that is, we usually have $\phi^*\circ d^*_{\bar{Q}}\neq d^*_Q\circ\phi^*$.

Example 2.23. Consider a rack Q. The inner automorphism group $\mathrm{Inn}(Q)$ acts naturally on Q. The set of orbits $\bar{Q}=Q/\mathrm{Inn}(Q)$ can be regarded as a trivial rack, in which case the quotient map $\phi\colon Q\to\bar{Q}$ becomes a rack homomorphism.

Consider a cochain $f\in C^n_{\mathrm{YB}}(c_{\bar{Q}};\mathfrak{m})$. The coboundary $d^*_{\bar{Q}}$ vanishes, so that $\phi^* d^*_{\bar{Q}} f=0$. In general, however, we have $d^*_Q\phi^* f\neq 0$. To see this consider $y,z\in Q$ satisfying $y\not\equiv z$, which means that there exists $x\in Q$ such that $x^y\neq x^z$. We find

$$\begin{aligned}\Big(d^1_Q(\phi^* f)\Big)\begin{bmatrix} x,y \\ x,z \end{bmatrix} &= (\phi^* f)\begin{bmatrix} y \\ z \end{bmatrix}\cdot\mathrm{id}\begin{bmatrix} x^y \\ x^z \end{bmatrix} - (\phi^* f)\begin{bmatrix} y \\ z \end{bmatrix}\cdot\mathrm{id}\begin{bmatrix} x \\ x \end{bmatrix} \\ &\quad - (\phi^* f)\begin{bmatrix} x \\ x \end{bmatrix}\cdot\mathrm{id}\begin{bmatrix} y \\ z \end{bmatrix} + (\phi^* f)\begin{bmatrix} x^y \\ x^z \end{bmatrix}\cdot\mathrm{id}\begin{bmatrix} y \\ z \end{bmatrix} = -f\begin{bmatrix} \phi(y) \\ \phi(z) \end{bmatrix}.\end{aligned}$$

This is in general not zero, whence $d^*_Q\phi^* f\neq\phi^* d^*_{\bar{Q}} f$. (For an explicit example take $\bar{Q}$ to be trivial, so that $d^*_{\bar{Q}}=0$ and f can be chosen arbitrarily.) We conclude that the natural candidate $\phi^*\colon C^*_{\mathrm{YB}}(c_{\bar{Q}};\mathfrak{m})\to C^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$ is not a cochain map.

3. DIAGONAL DEFORMATIONS

In §2 we have considered arbitrary deformations of c_Q. The problem becomes much easier if we concentrate on deformations of the form $\tilde{c}(a \otimes b) = \lambda(a,b) \cdot c_Q(a \otimes b)$, where $\lambda \colon Q \times Q \to \Lambda$ is a map into some abelian group Λ. Such deformations are classified by rack cohomology:

Definition 3.1. Let Q be a rack and let Λ be an abelian group (written additively). We consider the cochain complex $C^n_{\mathrm{R}} = C^n_{\mathrm{R}}(Q;\Lambda)$ formed by all maps $\lambda \colon Q^n \to \Lambda$. The coboundary $\delta^n \colon C^n_{\mathrm{R}} \to C^{n+1}_{\mathrm{R}}$ is defined by

$$(\delta^n \lambda)(a_0, \dots, a_n) = \sum_{i=1}^{n} (-1)^i \Big[\lambda(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) - \lambda(a_0^{a_i}, \dots, a_{i-1}^{a_i}, a_{i+1}, \dots, a_n) \Big]. \tag{3.1}$$

This defines a cochain complex $(C^*_{\mathrm{R}}, \delta^*)$, whose cohomology $H^n_{\mathrm{R}}(Q;\Lambda)$ is called the *rack cohomology* of Q with coefficients in Λ.

Remark 3.2. It is easily seen that $\tilde{c}$ is a Yang-Baxter operator if and only if λ is a rack cocycle, see Graña [26]. Likewise, $\tilde{c}$ and c_Q are equivalent if and only if λ is a coboundary.

Remark 3.3. As in group theory, the second cohomology group $H^2_{\mathrm{R}}(Q;\Lambda)$ is in bijective correspondence with equivalence classes of central extension $\Lambda \curvearrowright \tilde{Q} \to Q$, see [14, 15].

Let us make explicit how rack cohomology fits into the more general framework of Yang-Baxter cohomology. Suppose that Λ is a module over some ring $\mathbb{K}$. We can form the $\mathbb{K}$-algebra $\mathbb{A} = \mathbb{K} \oplus \Lambda$ by setting $uv = 0$ for all $u, v \in \Lambda$, that is, we equip $\mathbb{A}$ with the product $(a,u) \cdot (b,v) = (ab, av + bu)$. For $\mathbb{K} = \Lambda = \mathbb{Z}/_n$, for example, we obtain $\mathbb{A} = \mathbb{K}[h]/(h^2)$.

We have an augmentation homomorphism $\varepsilon \colon \mathbb{A} \to \mathbb{K}$ defined by $\varepsilon(1) = 1$ and $\varepsilon(u) = 0$ for all $u \in \Lambda$. The augmentation ideal $\mathfrak{m} = \ker(\varepsilon)$ thus coincides with Λ. Notice also that the additive group Λ is isomorphic to the multiplicative subgroup $1 + \mathfrak{m}$ of the ring $\mathbb{A}$.

If we consider diagonal deformations

$$\tilde{c}(a \otimes b) = \big(1 + \lambda(a,b)\big) \cdot c_Q(a \otimes b) \quad \text{with } \lambda(a,b) \in \mathfrak{m},$$

then we see that rack cohomology naturally embeds into Yang-Baxter cohomology:

Proposition 3.4. *The rack cochain complex $C^*_{\mathrm{R}}(Q;\Lambda)$ is naturally isomorphic to the diagonal subcomplex C^*_{Diag} of the Yang-Baxter cohomology $C^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$.*

Proof. As usual, a matrix $f \colon Q^n \times Q^n \to \mathfrak{m}$ is called *diagonal* if $f\left[\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right]$ vanishes whenever $x_i \neq y_i$ for some index $i = 1, \dots, n$. Diagonal cochains form a subcomplex of (C^*_{YB}, d^*).

Each diagonal matrix $f \colon Q^n \times Q^n \to \mathfrak{m}$ can be identified with the corresponding map $\lambda \colon Q^n \to \mathfrak{m}$ such that $f\left[\begin{smallmatrix} x_1, \dots, x_n \\ x_1, \dots, x_n \end{smallmatrix}\right] = \lambda(x_1, \dots, x_n)$. Under this identification, the Yang-Baxter coboundary (2.4) reduces to the rack coboundary (3.1). □

Remark 3.5. Unlike the full Yang-Baxter complex $(C^*_{\mathrm{YB}}(c_Q, \mathfrak{m}), d^*)$, the diagonal complex $(C^*_{\mathrm{Diag}}(c_Q, \mathfrak{m}), d^*)$ and rack cohomology (3.1) *are* functorial in Q.

Proposition 3.6. *There exists a retraction $r \colon C^*_{\mathrm{YB}} \to C^*_{\mathrm{Diag}}$ of cochain complexes, whence rack cohomology $H^*_{\mathrm{R}}(Q;\Lambda)$ is a direct summand of Yang-Baxter cohomology $H^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$.*

Proof. The obvious idea turns out to work. We define $r^n \colon C^n_{\mathrm{YB}} \to C^n_{\mathrm{Diag}}$ by

$$(r^n f)\left[\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right] := \begin{cases} f\left[\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right] & \text{if } x_i = y_i \text{ for all } i = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

The coboundary formula (2.4) shows that $d^n_i \circ r^n = r^{n+1} \circ d^n_i$, whence $d \circ r = r \circ d$. By construction we have $r(C^*_{\mathrm{YB}}) = C^*_{\mathrm{Diag}}$ and $r|C^*_{\mathrm{Diag}} = \mathrm{id}$, so that r is a retraction, as desired. □

Remark 3.7. The example of a trivial rack Q shows that Yang-Baxter $H^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$ is in general much bigger than rack cohomology $H^*_{\mathrm{R}}(Q;\Lambda)$, so we cannot capture all information by diagonal deformations alone. In order to do so, we have to consider the more general notion of *quasi-diagonal* deformations, as explained below.

4. Quasi-diagonal deformations

A matrix $f\colon Q^n \times Q^n \to \mathbb{A}$ is called *quasi-diagonal* if $f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right]$ vanishes whenever $x_i \not\equiv y_i$ for some index $i = 1,\dots,n$.

Proposition 4.1. *The quasi-diagonal cochains of the Yang-Baxter complex* (C^*_{YB}, d^*) *form a subcomplex, denoted* (C^*_Δ, d^*).

Remark 4.2. Restricted to the subcomplex C^*_Δ of quasi-diagonal cochains, the coboundary $d^n\colon C^n_\Delta \to C^{n+1}_\Delta$ takes the form $d^n f = \sum_{i=1}^n (-1)^i d^n_i f$ with

$$(d^n_i f)\begin{bmatrix} x_0,\dots,x_n \\ y_0,\dots,y_n \end{bmatrix} = \left(f\begin{bmatrix} x_0\,,\dots,x_{i-1},x_{i+1},\dots,x_n \\ y_0\,,\dots,y_{i-1},y_{i+1},\dots,y_n \end{bmatrix} - f\begin{bmatrix} x_0^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,x_n \\ y_0^{y_i},\dots,y_{i-1}^{y_i},y_{i+1},\dots,y_n \end{bmatrix} \right) \cdot \mathrm{id}\begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

This illustrates, in explicit terms, that the quasi-diagonal subcomplex is half-way between Yang-Baxter cohomology (2.4) and rack cohomology (3.1).

We should point out that the quasi-diagonal subcomplex C^*_Δ coincides with the Yang-Baxter cochain complex C^*_{YB} if the rack Q is trivial. On the other hand, C^*_Δ coincides with the rack cochain complex C^*_{R} if the inner representation $\rho\colon Q \to \mathrm{Inn}(Q)$ is injective: in this case $x \equiv y$ means $x = y$, and quasi-diagonal means diagonal.

Remark 4.3. Like the full Yang-Baxter complex $C^*_{\mathrm{YB}}(c_Q,\mathfrak{m})$, the quasi-diagonal complex $C^*_\Delta(c_Q,\mathfrak{m})$ is *not* functorial in the rack Q. Every rack homomorphism $\phi\colon Q \to \bar{Q}$ induces a map $\phi^*_\Delta\colon C_\Delta(c_{\bar{Q}},\mathfrak{m}) \to C_\Delta(c_Q,\mathfrak{m})$ defined by

$$(\phi^*_\Delta f)\begin{bmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{bmatrix} = f\begin{bmatrix} \phi(x_1),\dots,\phi(x_n) \\ \phi(y_1),\dots,\phi(y_n) \end{bmatrix} \tag{4.1}$$

for all $x_1 \equiv y_1,\dots,x_n \equiv y_n$ in Q. This natural map, however, is in general not a cochain map. A concrete example can be constructed as follows.

Example 4.4. Consider a non-trivial rack $\bar{Q}$ and choose $\bar{x},\bar{y} \in \bar{Q}$ such that $\bar{x}^{\bar{y}} \neq \bar{x}$. Assume that $\phi\colon Q \to \bar{Q}$ is a rack homomorphism, $\phi(x) = \bar{x}$ and $\phi(y) = \phi(z) = \bar{y}$ with $y \neq z$. The easiest example is the trivial extension $Q = \bar{Q} \times \{1,2\}$, which also ensures that $y = (\bar{y},1)$ and $z = (\bar{y},2)$ are behaviourally equivalent. For each cochain $f \in C^1(c_{\bar{Q}},\mathfrak{m})$ we find

$$(d^1\phi^* f)\begin{bmatrix} x,y \\ x,z \end{bmatrix} = \left((\phi^* f)\begin{bmatrix} x^y \\ x^z \end{bmatrix} - (\phi^* f)\begin{bmatrix} x \\ x \end{bmatrix} \right) \cdot \mathrm{id}\begin{bmatrix} y \\ z \end{bmatrix} = 0 \qquad \text{as opposed to} \tag{4.2}$$

$$(\phi^* d^1 f)\begin{bmatrix} x,y \\ x,z \end{bmatrix} = (d^1 f)\begin{bmatrix} \bar{x},\bar{y} \\ \bar{x},\bar{y} \end{bmatrix} = \left(f\begin{bmatrix} \bar{x}^{\bar{y}} \\ \bar{x}^{\bar{y}} \end{bmatrix} - f\begin{bmatrix} \bar{x} \\ \bar{x} \end{bmatrix} \right) \mathrm{id}\begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} = f\begin{bmatrix} \bar{x}^{\bar{y}} \\ \bar{x}^{\bar{y}} \end{bmatrix} - f\begin{bmatrix} \bar{x} \\ \bar{x} \end{bmatrix}. \tag{4.3}$$

Since $\bar{x}^{\bar{y}} \neq \bar{x}$, the cochain f can be so chosen that the last difference is non-zero.

The difference between equations (4.2) and (4.3) disappears for equivariant cochains:

Definition 4.5. A cochain $f \in C^n(c_Q,\mathfrak{m})$ is *fully equivariant* if it satisfies

$$f\begin{bmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{bmatrix} = f\begin{bmatrix} x_1^{g_1},\dots,x_n^{g_n} \\ y_1^{g_1},\dots,y_n^{g_n} \end{bmatrix}$$

for all $x_1,\dots,x_n,y_1,\dots,y_n \in Q$ and $g_1,\dots,g_n \in \mathrm{Inn}(Q)$.

Definition 4.6. A cochain $f \in C^n(c_Q,\mathfrak{m})$ is called *entropic* if it is quasi-diagonal and fully equivariant. Such cochains are characterized by the condition $d^n_0 f = \dots = d^n_n f = 0$, in other words, all partial coboundaries vanish [15, Lemma 30]. In particular, entropic cochains are cocycles; the submodule of entropic cocycles is denoted by $E^*(c_Q,\mathfrak{m}) \subset Z^*_{\mathrm{YB}}(c_Q,\mathfrak{m})$.

Remark 4.7. For every rack Q we have $C^*_{\mathrm{YB}}(c_Q,\mathfrak{m}) \supset C^*_\Delta(c_Q,\mathfrak{m}) \supset E^*(c_Q,\mathfrak{m})$ by definition. Every rack homomorphism $\phi\colon Q \to \bar{Q}$ induces maps

$$\begin{array}{ccccc} C^*_{\mathrm{YB}}(c_Q,\mathfrak{m}) & \xleftarrow{\supset} & C^*_\Delta(c_Q,\mathfrak{m}) & \xleftarrow{\supset} & E^*(c_Q,\mathfrak{m}) \\ {\scriptstyle\phi^*}\uparrow & & \uparrow{\scriptstyle\phi^*_\Delta} & & \uparrow{\scriptstyle\phi^*_E} \\ C^*_{\mathrm{YB}}(c_{\bar{Q}},\mathfrak{m}) & \xleftarrow{\supset} & C^*_\Delta(c_{\bar{Q}},\mathfrak{m}) & \xleftarrow{\supset} & E^*(c_{\bar{Q}},\mathfrak{m}). \end{array}$$

Here ϕ^* is defined by (2.8), whereas ϕ^*_Δ is defined by (4.1), and the map ϕ^*_E is obtained from ϕ^*_Δ by restriction. In general ϕ^* and ϕ^*_Δ are *not* cochain maps, as pointed out above. Only the third map ϕ^*_E is always a cochain map because $C^*_E \subset Z^*_{\mathrm{YB}}$ is a trivial subcomplex.

The main goal of this article is to show that $C^*_\Delta \subset C^*_{\mathrm{YB}}$ is a homotopy retract. We point out that a much weaker statement follows easily from the definition:

Proposition 4.8. *There exists a retraction $r\colon C^*_{\mathrm{YB}} \to C^*_\Delta$, whence quasi-diagonal Yang-Baxter cohomology $H^*_\Delta(c_Q;\mathfrak{m})$ is a direct summand of Yang-Baxter cohomology $H^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$.*

Proof. Again the obvious idea turns out to work. We define $r^n\colon C^n_{\mathrm{YB}} \to C^n_\Delta$ by

$$(r^n f)\left[\begin{smallmatrix}x_1,\dots,x_n\\ y_1,\dots,y_n\end{smallmatrix}\right] := \begin{cases} f\left[\begin{smallmatrix}x_1,\dots,x_n\\ y_1,\dots,y_n\end{smallmatrix}\right] & \text{if } x_i \equiv y_i \text{ for all } i=1,\dots,n, \\ 0 & \text{otherwise.}\end{cases}$$

The coboundary formula (2.4) shows that $d^n_i \circ r^n = r^{n+1} \circ d^n_i$, whence $d \circ r = r \circ d$. By construction we have $r(C^*_{\mathrm{YB}}) = C^*_\Delta$ and $r|C^*_\Delta = \mathrm{id}$, so r is a retraction, as desired. □

5. Constructing a homotopy retraction

Having set the scene in the preceding sections, we can now study the subcomplex $C^*_\Delta \subset C^*_{\mathrm{YB}}$ of quasi-diagonal cochains. It is easy to see that it is a retract, but it is more delicate to prove that it is in fact a homotopy retract. To this end we introduce an auxiliary filtration $C^*_{\mathrm{YB}} \supset C^*_1 \supset \cdots \supset C^*_\infty = C^*_\Delta$ of subcomplexes (§5.1) and prove that each complex homotopy retracts to its successor (§5.2). It then suffices to compose these piecewise homotopies in order to obtain the desired homotopy retraction from C^*_{YB} to C^*_Δ (§5.3).

5.1. An intermediate filtration.

We now turn to the problem of finding a homotopy retraction to the subcomplex $C^*_\Delta \subset C^*_{\mathrm{YB}}$ of quasi-diagonal cochains. The construction of Proposition 4.8 is nice and simple, but unfortunately the retraction $r\colon C^*_{\mathrm{YB}} \to C^*_\Delta$ does not seem to be a homotopy retraction, i.e. it is quite likely *not* homotopic to the identity of C^n_{YB}.

For reasons that will become clear in the following calculations, it is rather complicated to directly define a homotopy retraction to the subcomplex $C^*_\Delta \subset C^*_{\mathrm{YB}}$. The approach that we present here resolves this difficulty by induction on a judiciously chosen filtration

$$C^*_{\mathrm{YB}} = C^*_0 \supset C^*_1 \supset C^*_2 \supset \cdots \supset C^*_\infty = C^*_\Delta.$$

This will allow us to construct the homotopy retraction $C^*_{\mathrm{YB}} \twoheadrightarrow C^*_\Delta$ as the composition of partial retractions $p^*_m\colon C^*_m \twoheadrightarrow C^*_{m+1}$ which are much easier to understand. Figuratively speaking, we thus construct the deformation from C^*_{YB} to C^*_Δ by a piecewise linear path.

Definition 5.1. For each $m \in \mathbb{N}$ we define $C^*_m \subset C^*_{\mathrm{YB}}$ to be the subcomplex of cochains that are quasi-diagonal in the last m variables. More explicitly:

$$C^n_m := \left\{ f \in C^n_{\mathrm{YB}} \mid f\left[\begin{smallmatrix}x_1,\dots,x_n\\ y_1,\dots,y_n\end{smallmatrix}\right] = 0 \text{ if } x_i \not\equiv y_i \text{ for some index } i \text{ with } n-m < i \le n \right\}.$$

In each degree n we thus obtain a filtration $C^n_{\mathrm{YB}} = C^n_0 \supset C^n_1 \supset \cdots \supset C^n_n$ that stabilizes at C^n_n: obviously $C^n_m = C^n_n$ for all $m > n$. In each degree n its limit is thus $\bigcap_m C^n_m = C^n_n$.

Lemma 5.2. *The coboundary $d^n\colon C^n_{\mathrm{YB}} \to C^{n+1}_{\mathrm{YB}}$ satisfies $d^n(C^n_m) \subset C^{n+1}_m$ for each $m \in \mathbb{N}$. In other words, $(C^*_m, d^*|_{C^*_m})$ is a subcomplex of (C^*_{YB}, d^*).*

Proof. Suppose that $f \in C_m^n$. Formula (2.4) for the partial coboundary shows that $d_i^n f$ is again in C_m^{n+1}. The same thus holds for the coboundary $d^n f = \sum_{i=0}^n (-1)^i d_i^n f$. $\square$

Notation. We will suppress the explicit mention of the coboundary map and denote the complex (C_{YB}^*, d^*) simply by C_{YB}^*. Likewise we write C_m^* for the complex $(C_m^*, d^*|_{C_m^*})$.

5.2. **Cochain homotopies.** We wish to show that the inclusion $\iota_{m+1}^*\colon C_{m+1}^* \hookrightarrow C_m^*$ is a *homotopy retract*. To this end we shall construct a cochain map $p_m^*\colon C_m^* \twoheadrightarrow C_{m+1}^*$ such that $p_m^* \circ \iota_{m+1}^* = \mathrm{id}_{m+1}^*$ and a cochain homotopy $\iota_{m+1}^* \circ p_m^* \simeq \mathrm{id}_m^*\colon C_m^* \to C_m^*$. Such a projection p_m^* is called a *homotopy retraction* see [37, §II.2]. Recall that a *cochain homotopy* is a map $s_m^n\colon C_m^n \to C_m^{n-1}$ such that $p_m^n - \mathrm{id}_m^n = d^{n-1} \circ s_m^n + s_m^{n+1} \circ d^n$. In the sequel we will prefer the sign convention $d^{n-1} \circ s_m^n - s_m^{n+1} \circ d^n$, which is logically equivalent.

Remark 5.3. We call the set $\Delta = \{(x,y) \in Q^2 \mid x \equiv y\}$ the *quasi-diagonal*. On its complement $\Delta^c = \{(x,y) \in Q^2 \mid x \not\equiv y\}$ we choose a map $\psi\colon \Delta^c \to Q^2$, $(x,y) \mapsto (u,v)$ such that $u \neq v$ but $u^x = v^y$. It is easy to see that such a map exists: the inequivalence $x \not\equiv y$ means that the inner automorphisms $z \mapsto z * x$ and $z \mapsto z * y$ are different. This is equivalent to saying that their inverses $z \mapsto z \overline{*} x$ and $z \mapsto z \overline{*} y$ are different: there exists $z \in Q$ such that $u = z \overline{*} x$ differs from $v = z \overline{*} y$. In other words we have $u \neq v$ but $u^x = v^y$.

Definition 5.4. Fix $n, m \in \mathbb{N}$. For $m \geq n$ we define $s_m^n\colon C_m^n \to C_m^{n-1}$ to be the zero map. For $0 \leq m \leq n-1$ we set $k := n - m$ and define $s_m^n\colon C_m^n \to C_m^{n-1}$ as follows:

$$(s_m^n f)\left[\begin{smallmatrix} x_2,\dots,x_n \\ y_2,\dots,y_n \end{smallmatrix}\right] := \begin{cases} f\left[\begin{smallmatrix} x_2,\dots,x_{k-1},u,x_k,\dots,x_n \\ y_2,\dots,y_{k-1},v,y_k,\dots,y_n \end{smallmatrix}\right] & \text{if } x_k \not\equiv y_k, \text{ with } (u,v) = \psi(x_k,y_k), \\ 0 & \text{if } x_k \equiv y_k. \end{cases}$$

This induces a map $t_m^n := d^{n-1} \circ s_m^n - s_m^{n+1} \circ d^n\colon C_m^n \to C_m^n$.

Theorem 5.5. *The cochain map $p_m^n := \mathrm{id}_m^n - (-1)^{n-m} t_m^n\colon C_m^n \to C_m^n$ sends C_m^n to the subcomplex C_{m+1}^n and restricts to the identity on C_{m+1}^n. By construction, the maps id_m^* and p_m^* are homotopy equivalent, and thus $C_{m+1}^* \hookrightarrow C_m^*$ is a homotopy retract. The inclusion thus induces an isomorphism $H^*(C_{m+1}^*) \xrightarrow{\sim} H^*(C_m^*)$ on cohomology.*

Proof. We first have to check that p is a cochain map. This follows at once from its definition:

$$\begin{aligned} d^n \circ p_m^n &= d^n - (-1)^{n-m}\left[d^n d^{n-1} s_m^n - d^n s_m^{n+1} d^n\right], \\ p_m^{n+1} \circ d^n &= d^n + (-1)^{n-m}\left[d^n s_m^{n+1} d^n - s_m^{n+2} d^{n+1} d^n\right]. \end{aligned}$$

The two properties $p_m^n(C_m^n) \subset C_{m+1}^n$ and $p_m^n|C_{m+1}^n = \mathrm{id}_{m+1}^n$ will be established in the following two lemmas. The remaining statements are standard consequences of cochain homotopy [37, §II.2]. $\square$

Lemma 5.6. *The map t_m^n satisfies $(t_m^n f)\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right] = (-1)^k f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right]$ whenever $x_k \not\equiv y_k$.*

Proof. We recall our short-hand notation $k := n - m$. We will calculate $t_m^n\colon C_m^n \to C_m^n$ by making the individual terms $d_i^{n-1} \circ s_m^n$ and $s_m^{n+1} \circ d_i^n$ explicit for $i = 0, \dots, n$. Let $f \in C_m^n$ and assume $x_k \not\equiv y_k$. We shall distinguish the three cases $i \leq k-2$ and $i = k-1$ and $i \geq k$.

First case. For $i = 0, \dots, k-2$ we find:

$$\begin{aligned}
(d_i^{n-1} s_m^n f) \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} &= +(s_m^n f) \begin{bmatrix} x_1 & , \dots, x_i & , x_{i+2}, \dots, x_k, \dots, x_n \\ y_1 & , \dots, y_i & , y_{i+2}, \dots, y_k, \dots, y_n \end{bmatrix} \cdot \mathrm{id} \begin{bmatrix} x_{i+1}^{x_{i+2}\cdots x_n} \\ y_{i+1}^{y_{i+2}\cdots y_n} \end{bmatrix} \\
&\quad -(s_m^n f) \begin{bmatrix} x_1^{x_{i+1}}, \dots, x_i^{x_{i+1}}, x_{i+2}, \dots, x_k, \dots, x_n \\ y_1^{y_{i+1}}, \dots, y_i^{y_{i+1}}, y_{i+2}, \dots, y_k, \dots, y_n \end{bmatrix} \cdot \mathrm{id} \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} \\
&= +f \begin{bmatrix} x_1 & , \dots, x_i & , x_{i+2}, \dots, u, x_k, \dots, x_n \\ y_1 & , \dots, y_i & , y_{i+2}, \dots, v, y_k, \dots, y_n \end{bmatrix} \cdot \mathrm{id} \begin{bmatrix} x_{i+1}^{x_{i+2}\cdots x_n} \\ y_{i+1}^{y_{i+2}\cdots y_n} \end{bmatrix} \\
&\quad -f \begin{bmatrix} x_1^{x_{i+1}}, \dots, x_i^{x_{i+1}}, x_{i+2}, \dots, u, x_k, \dots, x_n \\ y_1^{y_{i+1}}, \dots, y_i^{y_{i+1}}, y_{i+2}, \dots, v, y_k, \dots, y_n \end{bmatrix} \cdot \mathrm{id} \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} \\
&= \quad (d_i^n f) \begin{bmatrix} x_1, \dots, x_{k-1}, u, x_k, \dots, x_n \\ y_1, \dots, y_{k-1}, v, y_k, \dots, y_n \end{bmatrix} \\
&= \quad (s_m^{n+1} d_i^n f) \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix}.
\end{aligned}$$

The third of these four equalities needs justification. We have to verify that

$$x_{i+1}^{x_{i+2}\cdots x_{k-1} x_k \cdots x_n} = y_{i+1}^{y_{i+2}\cdots y_{-1} y_k \cdots y_n}$$

is equivalent to

$$x_{i+1}^{x_{i+2}\cdots x_{k-1} u x_k \cdots x_n} = y_{i+1}^{y_{i+2}\cdots y_{k-1} v y_k \cdots y_n}.$$

We can assume that $x_j \equiv y_j$ for all $k < j \leq n$, otherwise the factors involving f vanish by our hypothesis $f \in C_m^n$. So we only have to show that

$$x_{i+1}^{x_{i+2}\cdots x_{k-1} x_k} = y_{i+1}^{y_{i+2}\cdots y_{-1} y_k}$$

is equivalent to

$$x_{i+1}^{x_{i+2}\cdots x_{k-1} u x_k} = y_{i+1}^{y_{i+2}\cdots y_{k-1} v y_k}.$$

This follows from $(a * u) * x_k = (a * x_k) * (u * x_k)$ and $(b * v) * y_k = (b * y_k) * (v * y_k)$, and our construction $(u, v) = \psi(x_k, y_k)$ ensures that $u * x_k = v * y_k$.

Second case. For $i = k-1$ we find:

$$\begin{aligned}
(d_{k-1}^{n-1} s_m^n f) \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} &= +(s_m^n f) \begin{bmatrix} x_1 & , \dots, x_{k-1}, x_{k+1}, \dots, x_n \\ y_1 & , \dots, y_{k-1}, y_{k+1}, \dots, y_n \end{bmatrix} \cdot \mathrm{id} \begin{bmatrix} x_k^{x_{k+1}\cdots x_n} \\ y_k^{y_{k+1}\cdots y_n} \end{bmatrix} \\
&\quad -(s_m^n f) \begin{bmatrix} x_1^{x_k}, \dots, x_{k-1}^{x_k}, x_{k+1}, \dots, x_n \\ y_1^{y_k}, \dots, y_{k-1}^{y_k}, y_{k+1}, \dots, y_n \end{bmatrix} \cdot \mathrm{id} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \\
&= \quad 0.
\end{aligned}$$

The first factors vanish whenever $x_j \not\equiv y_j$ for some j with $k < j \leq n$; otherwise the second factors vanish because of our hypothesis $x_k \neq y_k$. On the other hand we have:

$$\begin{aligned}
(s_m^{n+1} d_{k-1}^n f) \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} &= +(d_{k-1}^n f) \begin{bmatrix} x_1, \dots, x_{k-1}, u, x_k, \dots, x_n \\ y_1, \dots, y_{k-1}, v, y_k, \dots, y_n \end{bmatrix} \\
&= +f \begin{bmatrix} x_1, \dots, x_{k-1}, x_k, \dots, x_n \\ y_1, \dots, y_{k-1}, y_k, \dots, y_n \end{bmatrix} \cdot \mathrm{id} \begin{bmatrix} u^{x_k \cdots x_n} \\ v^{y_k \cdots y_n} \end{bmatrix} \\
&\quad -f \begin{bmatrix} x_1^u, \dots, x_{k-1}^u, x_k, \dots, x_n \\ y_1^v, \dots, y_{k-1}^v, y_k, \dots, y_n \end{bmatrix} \cdot \mathrm{id} \begin{bmatrix} u \\ v \end{bmatrix} \\
&= \quad f \begin{bmatrix} x_1, \dots, x_{k-1}, x_k, \dots, x_n \\ y_1, \dots, y_{k-1}, y_k, \dots, y_n \end{bmatrix}.
\end{aligned}$$

The first factors vanish whenever $x_j \not\equiv y_j$ for some j with $k < j \leq n$; otherwise we have $u \neq v$ with $u^{x_k} = v^{y_k}$, whence $u^{x_k \cdots x_n} = v^{y_k \cdots y_n}$.

Third case. For $i \geq k$ we find:

$$\begin{aligned}
(d_i^{n-1} s_m^n f)\begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} &= +(s_m^n f)\begin{bmatrix} x_1 & ,\dots, x_k & ,\dots, x_i & , x_{i+2}, \dots, x_n \\ y_1 & ,\dots, y_k & ,\dots, y_i & , y_{i+2}, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1}^{x_{i+2}\cdots x_n} \\ y_{i+1}^{y_{i+2}\cdots y_n} \end{bmatrix} \\
&\quad -(s_m^n f)\begin{bmatrix} x_1^{x_{i+1}}, \dots, x_k^{x_{i+1}}, \dots, x_i^{x_{i+1}}, x_{i+2}, \dots, x_n \\ y_1^{y_{i+1}}, \dots, y_k^{y_{i+1}}, \dots, y_i^{y_{i+1}}, y_{i+2}, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} \\
&= \quad 0.
\end{aligned}$$

The first summand vanishes because $x_k \not\equiv y_k$; the second summand vanishes because $x_{i+1} \neq y_{i+1}$ or $x_k^{x_{i+1}} \not\equiv y_k^{y_{i+1}}$. Analogously:

$$\begin{aligned}
(s_m^{n+1} d_k^n f)\begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} &= +(d_k^n f)\begin{bmatrix} x_1, \dots, x_{k-1}, u, x_k, \dots, x_n \\ y_1, \dots, y_{k-1}, v, y_k, \dots, y_n \end{bmatrix} \\
&= +f\begin{bmatrix} x_1 & ,\dots, x_{k-1}, u & , x_{k+1}, \dots, x_n \\ y_1 & ,\dots, y_{k-1}, v & , y_{k+1}, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_k^{x_{k+1}\cdots x_n} \\ y_k^{y_{k+1}\cdots y_n} \end{bmatrix} \\
&\quad -f\begin{bmatrix} x_1^{x_k}, \dots, x_{k-1}^{x_k}, u^{x_k}, x_{k+1}, \dots, x_n \\ y_1^{y_k}, \dots, y_{k-1}^{y_k}, v^{y_k}, y_{k+1}, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_k \\ y_k \end{bmatrix} \\
&= \quad 0.
\end{aligned}$$

The first factors vanish whenever $x_j \not\equiv y_j$ for some j with $k < j \leq n$; otherwise the second factors vanish because of our hypothesis $x_k \neq y_k$. The same conclusion holds for $i > k$:

$$\begin{aligned}
(s_m^{n+1} d_i^n f)\begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} &= +(d_i^n f)\begin{bmatrix} x_1, \dots, x_{k-1}, u, x_k, \dots, x_n \\ y_1, \dots, y_{k-1}, v, y_k, \dots, y_n \end{bmatrix} \\
&= +f\begin{bmatrix} x_1, \dots, x_{k-1}, u & , x_k, \dots, x_{i-1}, x_{i+1}, \dots, x_n \\ y_1, \dots, y_{k-1}, v & , y_k, \dots, y_{i-1}, y_{i+1}, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_i^{x_{i+1}\cdots x_n} \\ y_i^{y_{i+1}\cdots y_n} \end{bmatrix} \\
&\quad -f\begin{bmatrix} x_1^{x_i}, \dots, x_{k-1}^{x_i}, u^{x_i}, x_k^{x_i}, \dots, x_{i-1}^{x_i}, x_{i+1}, \dots, x_n \\ y_1^{y_i}, \dots, y_{k-1}^{y_i}, v^{y_i}, y_k^{y_i}, \dots, y_{i-1}^{y_i}, y_{i+1}, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_i \\ y_i \end{bmatrix} \\
&= \quad 0.
\end{aligned}$$

The first summand vanishes because $x_k \not\equiv y_k$; the second summand vanishes because $x_i \neq y_i$ or $x_k^{x_i} \not\equiv y_k^{y_i}$. □

Lemma 5.7. *The map t_m^n satisfies $t_m^n f = 0$ whenever $f \in C_{m+1}^n$.*

Proof. We show that $(t_m^n f)\left[\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right] = 0$ for all $f \in C_{m+1}^n$ and all $x_1, \dots, x_n, y_1, \dots, y_n \in Q$. The previous lemma resolves the case $x_k \not\equiv y_k$, so it suffices to consider the remaining case where $x_k \equiv y_k$. By definition of s_m^{n+1} we have $(s_m^{n+1} d_i^n f)\left[\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right] = 0$ because $x_k \equiv y_k$. Likewise, for $i \leq k-2$ we find:

$$\begin{aligned}
(d_i^{n-1} s_m^n f)\begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} &= +(s_m^n f)\begin{bmatrix} x_1 & ,\dots, x_i & , x_{i+2}, \dots, x_k, \dots, x_n \\ y_1 & ,\dots, y_i & , y_{i+2}, \dots, y_k, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1}^{x_{i+2}\cdots x_n} \\ y_{i+1}^{y_{i+2}\cdots y_n} \end{bmatrix} \\
&\quad -(s_m^n f)\begin{bmatrix} x_1^{x_{i+1}}, \dots, x_i^{x_{i+1}}, x_{i+2}, \dots, x_k, \dots, x_n \\ y_1^{y_{i+1}}, \dots, y_i^{y_{i+1}}, y_{i+2}, \dots, y_k, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} \\
&= \quad 0.
\end{aligned}$$

For $i \geq k-1$, however, we find:

$$\begin{aligned}(d_i^{n-1} s_m^n f)\begin{bmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{bmatrix} &= +(s_m^n f)\begin{bmatrix} x_1 & ,\dots,x_{k-1},\dots,x_i & ,x_{i+2},\dots,x_n \\ y_1 & ,\dots,y_{k-1},\dots,y_i & ,y_{i+2},\dots,y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1}^{x_{i+2}\cdots x_n} \\ y_{i+1}^{y_{i+2}\cdots y_n} \end{bmatrix} \\ &\quad -(s_m^n f)\begin{bmatrix} x_1^{x_{i+1}},\dots,x_{k-1}^{x_{i+1}},\dots,x_i^{x_{i+1}},x_{i+2},\dots,x_n \\ y_1^{y_{i+1}},\dots,y_{k-1}^{y_{i+1}},\dots,y_i^{y_{i+1}},y_{i+2},\dots,y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix}.\end{aligned}$$

The summands are non-zero only if $x_{i+1} = y_{i+1}$ and $x_j \equiv y_j$ for all j with $k \leq j \leq n$: in this case their difference measures the defect of $s_m^n f$ to being equivariant (jointly in the first i variables). Both summands vanish if $x_{k-1} \equiv y_{k-1}$, so let us assume $x_{k-1} \not\equiv y_{k-1}$:

$$\begin{aligned}(d_i^{n-1} s_m^n f)\begin{bmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{bmatrix} &= +f\begin{bmatrix} x_1 & ,\dots,u' & ,x_{k-1},\dots,x_i & ,x_{i+2},\dots,x_n \\ y_1 & ,\dots,v' & ,y_{k-1},\dots,y_i & ,y_{i+2},\dots,y_n \end{bmatrix} \\ &\quad -f\begin{bmatrix} x_1^{x_{i+1}},\dots,u'',x_{k-1}^{x_{i+1}},\dots,x_i^{x_{i+1}},x_{i+2},\dots,x_n \\ y_1^{y_{i+1}},\dots,v'',y_{k-1}^{y_{i+1}},\dots,y_i^{y_{i+1}},y_{i+2},\dots,y_n \end{bmatrix}\end{aligned} \tag{5.1}$$

Here $(u',v') = \psi(x_{k-1},y_{k-1})$ and $(u'',v'') = \psi(x_{k-1}^{x_{i+1}},y_{k-1}^{y_{i+1}})$. For $f \in C_m^n$ the contributions do in general not cancel. We see, however, that both summands vanish if $f \in C_{m+1}^n$. □

Remark 5.8. Equation (5.1) shows that $(t_m^n f)\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right]$ can be non-zero for $f \in C_m^n$, if $x_k \equiv y_k$ but $x_{k-1} \not\equiv y_{k-1}$. This equation measures the defect of the cochain f, and our auxiliary map $\psi\colon (x_{k-1},y_{k-1}) \to (u,v)$, to be equivariant under the action of $|\mathrm{Inn}(Q)|$. In the equivariant setting of [15] this defect disappears, and the projection p_m^n becomes

$$(p_m^n f)\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right] := \begin{cases} 0 & \text{if } x_j \not\equiv y_j \text{ for some } j \text{ with } n-m \leq j \leq n, \\ f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right] & \text{otherwise.} \end{cases}$$

This simplified formula has been used in [15], where symmetrization was applied throughout to simplify calculations. In our present setting we cannot apply symmetrization and thus cannot assume equivariance. It is remarkable, therefore, that the above calculations carry through. The price to pay is that the projection p_m^n has a more complicated form.

5.3. **Composition of homotopy retractions.** Having constructed homotopy retractions $C_0^* \twoheadrightarrow C_1^* \twoheadrightarrow \dots \twoheadrightarrow C_{m-1}^* \twoheadrightarrow C_m^*$ in §5.2, it now suffices to put the pieces together:

Corollary 5.9. *The subcomplex C_Δ^* of quasi-diagonal cochains is a homotopy retract of the full Yang-Baxter cochain complex C_{YB}^*. As a consequence the inclusion $C_\Delta^* \hookrightarrow C_{\mathrm{YB}}^*$ induces an isomorphism on cohomology, $H^*(C_\Delta^*) \xrightarrow{\sim} H^*(C_{\mathrm{YB}}^*)$.*

Proof. The composition of homotopic cochain maps yields again homotopic cochain maps. As a consequence, the composition of our partial homotopy retractions yields again a homotopy retraction

$$P_m^* := p_{m-1}^* \circ p_{m-2}^* \circ \dots \circ p_1^* \circ p_0^* \colon C_0^* \to C_m^*.$$

This shows that the inclusion $C_m^* \hookrightarrow C_{\mathrm{YB}}^*$ is a homotopy retract. We wish to pass to the limit $C_\Delta^* = \bigcap_m C_m^*$. In each degree n we have $p_m^n = \mathrm{id}_n^n$ for all $m \geq n$, and thus $P_m^n = P_n^n$. We can thus define $P_\infty^* = \lim_{m\to\infty} P_m^*$ as the degree-wise limit $P_\infty^n = P_n^n$. We conclude that $C_\Delta^* \hookrightarrow C_{\mathrm{YB}}^*$ is a homotopy retract. □

6. From infinitesimal to complete deformations

In this section we will pass from infinitesimal to complete deformations. In order to do so, we will assume that the ring $\mathbb{A}$ is complete with respect to the ideal $\mathfrak{m}$, that is, we assume that the natural map $\mathbb{A} \to \varprojlim \mathbb{A}/\mathfrak{m}^n$ is an isomorphism.

Example 6.1. A polynomial ring $\mathbb{K}[h]$ is not complete with respect to the ideal (h). Its completion is the power series ring $\mathbb{K}[[h]] = \varprojlim \mathbb{K}[h]/(h^n)$. The latter is complete with respect to its ideal $\mathfrak{m} = (h)$. If $\mathbb{K}$ is a field, then $\mathbb{K}[[h]]$ is a complete local ring, which means that $\mathfrak{m}$ is the unique maximal ideal and $\mathbb{K}[[h]]$ is complete with respect to $\mathfrak{m}$.

Example 6.2. The ring of integers $\mathbb{Z}$ is not complete with respect to the ideal (p), where p will be assumed to be prime. Its completion is the ring of p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/_{p^n}$. The latter is complete with respect to its unique maximal ideal $\mathfrak{m} = (p)$.

Completions lend themselves to induction techniques. As the inductive step, we assume that $\mathfrak{m}^{n+1} = 0$. One can always force this condition by passing to the quotient $\mathbb{A}/\mathfrak{m}^{n+1}$.

Lemma 6.3. *Consider a ring $\mathbb{A}$ with ideal $\mathfrak{m}$ such that $\mathfrak{m}^{n+1} = 0$. Let $c\colon \mathbb{A}Q^2 \to \mathbb{A}Q^2$ be a Yang-Baxter operator that satisfies $c \equiv c_Q$ modulo $\mathfrak{m}$ and is quasi-diagonal modulo $\mathfrak{m}^n$. Then there exists $\alpha\colon \mathbb{A}Q \to \mathbb{A}Q$ with $\alpha \equiv \mathrm{id}_V$ modulo $\mathfrak{m}^n$, such that $(\alpha\otimes\alpha)^{-1}\circ c\circ(\alpha\otimes\alpha)$ is a quasi-diagonal deformation of c_Q.*

Proof. We have $c = c_Q \circ F$ with $F \equiv \mathrm{id}_V^{\otimes 2}$ modulo $\mathfrak{m}$. We write F in matrix notation as a map $F\colon Q^2 \times Q^2 \to \mathbb{A}$. Its non-quasi-diagonal part $f\colon Q^2\times Q^2 \to \mathbb{A}$ is defined by

$$f\left[\begin{smallmatrix} x_1, x_2 \\ y_1, y_2 \end{smallmatrix}\right] := \begin{cases} 0 & \text{if } x_1 \equiv y_1 \text{ and } x_2 \equiv y_2, \\ F\left[\begin{smallmatrix} x_1, x_2 \\ y_1, y_2 \end{smallmatrix}\right] & \text{otherwise.} \end{cases}$$

By hypothesis f takes values in $\mathfrak{m}^n \subset \mathbb{A}$, and can thus be considered as a cochain $C^2_{\mathrm{YB}}(c_Q;\mathfrak{m}^n)$. The map $\bar{c} = c_Q \circ (F - f) = c \circ (\mathrm{id}_V^{\otimes 2} - f)$ is quasi-diagonal, by construction.

We claim that $\bar{c}$ is actually a Yang-Baxter operator. We know that c satisfies the Yang-Baxter equation; its deformation $\bar{c}$ thus satisfies

$$\mathrm{id}_V^{\otimes 3} - (\mathrm{id}_V \otimes \bar{c})^{-1}(\bar{c}\otimes \mathrm{id}_V)^{-1}(\mathrm{id}_V \otimes \bar{c})^{-1}(\bar{c}\otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \bar{c})(\bar{c}\otimes \mathrm{id}_V) = d^2 f. \tag{6.1}$$

It is easy to check that the left-hand side is a quasi-diagonal map, whereas the right-hand side is zero on the quasi-diagonal. We conclude that *both* sides must vanish. This means that $\bar{c}$ satisfies the Yang-Baxter equation, and that $f \in C^2_{\mathrm{YB}}(c_Q;\mathfrak{m}^n)$ is a cocycle.

By Theorem 1.3, the inclusion $C^*_\Delta(c_Q;\mathfrak{m}^n) \subset C^*_{\mathrm{YB}}(c_Q;\mathfrak{m}^n)$ induces an isomorphism on cohomology. The class $[f] \in C^2_{\mathrm{YB}}(c_Q;\mathfrak{m}^n)$ can thus be presented by a quasi-diagonal cocycle $\tilde{f} \in C^2_\Delta(c_Q;\mathfrak{m}^n)$. This means that there exists a cochain $g \in C^1_{\mathrm{YB}}(c_Q;\mathfrak{m}^n)$ such that $\tilde{f} = f + d^1 g$. We conclude that $\alpha = \mathrm{id}_V + g$ conjugates c to a quasi-diagonal Yang-Baxter operator $\tilde{c} = (\alpha\otimes\alpha)^{-1}\circ c\circ(\alpha\otimes\alpha)$, as desired. □

Remark 6.4. In the preceding proof the construction and analysis of $\bar{c}$ serve to show that f is a 2-cocycle. The separation trick of Equation (6.1) is taken from [15, §4]. I seize the opportunity to point out that there the difference (6.1) is misprinted and lacks the term $\mathrm{id}_V^{\otimes 3}$. With this small correction the argument applies as intended.

Remark 6.5. In the proof of Lemma 6.3 we do not claim that c is conjugate to $\bar{c}$. This is true in the equivariant setting of [15], but without equivariance it is false in general: the coboundary $d^1 g$ kills the non-quasi-diagonal part but usually also changes the quasi-diagonal part (see Remark 5.8).

Theorem 6.6. *Let $\mathbb{A}$ be a ring that is complete with respect to the ideal $\mathfrak{m}$. Then every Yang-Baxter deformation of c_Q over $\mathbb{A}$ is equivalent to a quasi-diagonal deformation.*

Proof. Starting with $c_1 := c$ for $n = 1$, suppose that $c_n = c_Q f_n$ has a deformation term f_n that is quasi-diagonal modulo $\mathfrak{m}^n$. By Lemma 6.3, there exists $\alpha_n\colon \mathbb{A}Q \to \mathbb{A}Q$ with $\alpha_n \equiv \mathrm{id}_V$ modulo $\mathfrak{m}^n$, such that $c_{n+1} := (\alpha_n\otimes\alpha_n)^{-1} c_n (\alpha_n\otimes\alpha_n)$ is given by $c_{n+1} = c_Q f_{n+1}$ with f_{n+1} quasi-diagonal modulo $\mathfrak{m}^{n+1}$. The lemma ensures that such a map $\bar{\alpha}_n$ exists modulo $\mathfrak{m}^{n+1}$; this can be lifted to a map $\alpha_n\colon \mathbb{A}Q \to \mathbb{A}Q$, which is invertible because $\mathbb{A}$ is complete. Completeness of $\mathbb{A}$ also ensures that we can pass to the limit and define the infinite product $\alpha = \alpha_1\alpha_2\alpha_3\cdots$: for each $n \in \mathbb{N}$ this product is finite modulo $\mathfrak{m}^n$. By construction, $(\alpha\otimes\alpha)^{-1} c (\alpha\otimes\alpha)$ is quasi-diagonal and equivalent to c, as desired. □

Corollary 6.7. *If $H^2_{\mathrm{YB}}(c_Q;\mathfrak{m}/\mathfrak{m}^2) = \mathfrak{m}/\mathfrak{m}^2$, then c_Q is rigid over $(\mathbb{A},\mathfrak{m})$.*

Proof. For every unit $u \in 1+\mathfrak{m}$ we obtain a trivially deformed Yang-Baxter operator $\tilde{c} = u \cdot c_Q$. On the cochain level this corresponds to a constant multiple of the identity, which induces an injection $\mathfrak{m}/\mathfrak{m}^2 \hookrightarrow H^2_{\mathrm{YB}}(c_Q;\mathfrak{m}/\mathfrak{m}^2)$. If these trivial classes exhaust all cohomology classes, then degree-wise elimination as in the preceding proof conjugates any given Yang-Baxter deformation of c_Q to one of the form $u \cdot c_Q$. $\square$

7. Examples and applications

Example 7.1 (trivial quandle)**.** Consider first a trivial quandle Q, with $x * y = x$ for all x, y, so that $c_Q = \tau$ is simply the transposition operator. Here our results cannot add anything new, because the Yang-Baxter complex C^*_{YB} is trivial, i.e. $df = 0$ for all $f \in C^*_{\mathrm{YB}}$. In particular there are no infinitesimal obstructions: *every* deformation of τ satisfies the Yang-Baxter equation modulo $\mathfrak{m}^2$. There are, however, higher-order obstructions: these form a subject of their own and belong to the much deeper theory of quantum invariants [11, 40, 31, 32].

Example 7.2 (faithful quandle)**.** Next we consider the other extreme, where Theorem 1.1 applies most efficiently. Let G be a centreless group, so that conjugation induces an isomorphism $G \xrightarrow{\sim} \mathrm{Inn}(G)$. Suppose that $Q \subset G$ is a conjugacy class that generates G. Then we have $\mathrm{Inn}(Q) \cong \mathrm{Inn}(G) \cong G$, and the inner representation $\rho\colon Q \to \mathrm{Inn}(Q)$ is injective. In this case every Yang-Baxter deformation of c_Q over a complete ring $\mathbb{A}$ is equivalent to a diagonal deformation. If $|G|$ is finite and invertible in $\mathbb{A}$, then c_Q is rigid [15].

Example 7.3 (dihedral quandle of order 3)**.** The smallest non-trivial example of a rigid operator c_Q is given by the quandle $Q = \{(12),(13),(23)\}$, formed by transpositions in the symmetric group S_3, or equivalently the set of reflections of an equilateral triangle.

Ordering the basis $Q \times Q$ lexicographically, we can represent c_Q by the matrix

$$c_Q = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}.$$

In the quantum case, the initial operator τ is trivial but its deformations are highly interesting. In the present example, the interesting part is the initial operator c_Q itself: the associated link invariant is the number of 3-colourings, as defined by R.H. Fox [22, 23].

Unlike τ, the operator c_Q does not admit any non-trivial deformation over $\mathbb{Q}[\![h]\!]$. In this sense it is an isolated solution of the Yang-Baxter equation. We can now prove more:

Proposition 7.4. *For the quandle $Q = \{(12),(13),(23)\} \subset S_3$ the associated Yang-Baxter operator c_Q is rigid over every complete ring.*

Proof. According to [15], the operator c_Q is rigid over every ring $\mathbb{A}$ in which the order $|S_3| = 6$ is invertible. Potentially there could exist non-trivial deformations in characteristic 2 or 3. Theorem 1.3 ensures that infinitesimal deformations are quasi-diagonal, which means diagonal in the present example because $\rho\colon Q \to \mathrm{Inn}(Q) = S_3$ is injective. According to Proposition 3.4, diagonal deformations correspond to rack cohomology. A direct calculation shows that $H^2_{\mathrm{R}}(Q;\mathfrak{m}) \cong \mathfrak{m}$ for all modules $\mathfrak{m}$, see [9], whence Corollary 6.7 implies rigidity. $\square$

Example 7.5 (the other quandle of order 3)**.** The smallest quandle that is non-trivial yet deformable is $Q = \{a, b, c\}$ with operation given by the table below. Ordering the basis $Q \times Q$ lexicographically, we obtain the matrix of c_Q as indicated. We have $\mathrm{Inn}(Q) = \mathbb{Z}/2$: if 2 is invertible in $\mathbb{K}$, then $H^2_{\mathrm{YB}}(c_Q;\mathbb{K})$ is free of rang 9 and can easily be made explicit using the results of [15]. We state it here in form of a 9-parameter deformation $c = c_Q \circ (\mathrm{id}_V^{\otimes 2} + f)$,

where $f \in C^2_{\mathrm{YB}}(c_Q, \mathfrak{m})$ is quasi-diagonal and equivariant under $\mathrm{Inn}(Q) \times \mathrm{Inn}(Q)$:

$$Q = \left\{ \begin{array}{c|ccc} * & a & b & c \\ \hline a & a & a & b \\ b & b & b & a \\ c & c & c & c \end{array} \right. , \quad c_Q = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad f = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdot & \lambda_3 & \lambda_4 & \cdot & \cdot & \cdot & \cdot \\ \lambda_2 & \lambda_1 & \cdot & \lambda_4 & \lambda_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \lambda_5 & \cdot & \cdot & \lambda_6 & \cdot & \cdot & \cdot \\ \lambda_3 & \lambda_4 & \cdot & \lambda_1 & \lambda_2 & \cdot & \cdot & \cdot & \cdot \\ \lambda_4 & \lambda_3 & \cdot & \lambda_2 & \lambda_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \lambda_6 & \cdot & \cdot & \lambda_5 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_7 & \lambda_8 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_8 & \lambda_7 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_9 \end{pmatrix}.$$

We remark that the deformed operator c satisfies the Yang-Baxter equation to all orders, and not only infinitesimally modulo $\mathfrak{m}^2$.

A priori there could exist more deformations over $\mathbb{Z}/_2$, but a computer calculation shows that $\dim H^2_{\mathrm{YB}}(c_Q; \mathbb{Z}/_2) = 9$. So there are no additional deformations in the modular case.

Example 7.6 (dihedral quandle of order 4)**.** There exist quandles for which the modular case offers more Yang-Baxter deformations than the rational case. We wish to illustrate this by an example where the additional deformations are not diagonal but quasi-diagonal. The smallest such example is given by the set of reflections of a square,

$$Q = \{\, (13),\ (24),\ (12)(34),\ (14)(23) \,\}.$$

This set is closed under conjugation in the symmetric group S_4, hence a quandle. With respect to the lexicographical basis, c_Q is represented by the following permutation matrix:

$$c_Q = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}.$$

By construction, this matrix is a solution of the Yang-Baxter equation over any ring $\mathbb{A}$. According to [15] it admits a 16-parameter deformation $c(\lambda) = c_Q \circ (\mathrm{id}_V^{\otimes 2} + f)$ given by the following matrix, which is quasi-diagonal and equivariant under $\mathrm{Inn}(Q) \times \mathrm{Inn}(Q)$:

$$f = \begin{pmatrix}
\lambda_1 & \lambda_2 & \cdot & \cdot & \lambda_3 & \lambda_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\lambda_2 & \lambda_1 & \cdot & \cdot & \lambda_4 & \lambda_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_5 & \lambda_6 & \cdot & \cdot & \lambda_7 & \lambda_8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_6 & \lambda_5 & \cdot & \cdot & \lambda_8 & \lambda_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\lambda_3 & \lambda_4 & \cdot & \cdot & \lambda_1 & \lambda_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\lambda_4 & \lambda_3 & \cdot & \cdot & \lambda_2 & \lambda_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_7 & \lambda_8 & \cdot & \cdot & \lambda_5 & \lambda_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_8 & \lambda_7 & \cdot & \cdot & \lambda_6 & \lambda_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_9 & \lambda_{10} & \cdot & \cdot & \lambda_{11} & \lambda_{12} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{10} & \lambda_9 & \cdot & \cdot & \lambda_{12} & \lambda_{11} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{13} & \lambda_{14} & \cdot & \cdot & \lambda_{15} & \lambda_{16} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{14} & \lambda_{13} & \cdot & \cdot & \lambda_{16} & \lambda_{15} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{11} & \lambda_{12} & \cdot & \cdot & \lambda_9 & \lambda_{10} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{12} & \lambda_{11} & \cdot & \cdot & \lambda_{10} & \lambda_9 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{15} & \lambda_{16} & \cdot & \cdot & \lambda_{13} & \lambda_{14} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{16} & \lambda_{15} & \cdot & \cdot & \lambda_{14} & \lambda_{13}
\end{pmatrix}.$$

For every choice of parameters $\lambda_1, \dots, \lambda_{16} \in \mathfrak{m}$ the matrix $c(\lambda)$ satisfies the Yang-Baxter equation (to all orders) and thus deforms $c(0) = c_Q$ over $(\mathbb{A}, \mathfrak{m})$.

The quandle Q has the inner automorphism group $\mathrm{Inn}(Q) \cong \mathbb{Z}/_2 \times \mathbb{Z}/_2$, of order 4. If 2 is invertible in $\mathbb{K}$, then $H^*_{\mathrm{YB}}(c_Q; \mathbb{K})$ can be calculated using the results of [15] and is easily seen to be free of rank 16 such that f is the most general deformation. In particular we have $\dim H^*_{\mathrm{YB}}(c_Q; \mathbb{K}) = 16$ for every field $\mathbb{K}$ of characteristic $\neq 2$.

Over $\mathbb{K} = \mathbb{Z}/_2$, however, a computer calculation shows that $\dim H^2_{\mathrm{YB}}(c_Q; \mathbb{Z}/_2) = 20$, which means that there exists a 20-parameter deformation, at least infinitesimally. We state the result in the form $c = c_Q(\mathrm{id}_V^{\otimes 2} + f + g)$ as follows.

First we have the 16-parameter family that appears in every characteristic:

$$f = \begin{pmatrix}
\lambda_1 & \lambda_2 & \cdot & \cdot & \lambda_3 & \lambda_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\lambda_2 & \lambda_1 & \cdot & \cdot & \lambda_4 & \lambda_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_5' & \lambda_6 & \cdot & \cdot & \lambda_7' & \lambda_8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_6 & \lambda_5' & \cdot & \cdot & \lambda_8 & \lambda_7' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\lambda_3 & \lambda_4 & \cdot & \cdot & \lambda_1 & \lambda_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\lambda_4 & \lambda_3 & \cdot & \cdot & \lambda_2 & \lambda_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_7'' & \lambda_8 & \cdot & \cdot & \lambda_5'' & \lambda_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_8 & \lambda_7'' & \cdot & \cdot & \lambda_6 & \lambda_5'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_9' & \lambda_{10} & \cdot & \cdot & \lambda_{11}' & \lambda_{12} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{10} & \lambda_9' & \cdot & \cdot & \lambda_{12} & \lambda_{11}' & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{13} & \lambda_{14} & \cdot & \cdot & \lambda_{15} & \lambda_{16} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{14} & \lambda_{13} & \cdot & \cdot & \lambda_{16} & \lambda_{15} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{11}'' & \lambda_{12} & \cdot & \cdot & \lambda_9'' & \lambda_{10} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{12} & \lambda_{11}'' & \cdot & \cdot & \lambda_{10} & \lambda_9'' & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{15} & \lambda_{16} & \cdot & \cdot & \lambda_{13} & \lambda_{14} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{16} & \lambda_{15} & \cdot & \cdot & \lambda_{14} & \lambda_{13}
\end{pmatrix}.$$

For every choice of parameters in $\mathfrak{m}$ the matrix $c(\lambda) = c_Q \circ (\mathrm{id}_V^{\otimes 2} + f)$ satisfies the Yang-Baxter equation modulo $\mathfrak{m}^2$. It even satisfies the Yang-Baxter equation to any order provided that $\lambda_5' = \lambda_5''$, $\lambda_7' = \lambda_7''$, $\lambda_9' = \lambda_9''$, $\lambda_{11}' = \lambda_{11}''$.

We set $\lambda_5 = \lambda_5' + \lambda_5''$, $\lambda_7 = \lambda_7' + \lambda_7''$, $\lambda_9 = \lambda_9' + \lambda_9''$, $\lambda_{11} = \lambda_{11}' + \lambda_{11}''$. Two deformations $c(\lambda)$ and $c(\tilde{\lambda})$ are gauge equivalent if and only if $\lambda_k = \tilde{\lambda}_k$ for all $k = 1, \dots, 16$. We have chosen the redundant formulation above in order to highlight the symmetry resp. the symmetry breaking. If 2 were invertible we would simply set $\lambda_5' = \lambda_5'' = \frac{1}{2}\lambda_5$ etc. In characteristic 2 we can realize $\lambda_5 = 1$ either by $\lambda_5' = 1$ and $\lambda_5'' = 0$, or by $\lambda_5' = 0$ and $\lambda_5'' = 1$. Both deformations are gauge equivalent, but no symmetric form is possible.

Next we have 4-parameters deformation that appears only in characteristic 2:

$$g = \begin{pmatrix}
\alpha' & \cdot & \cdot & \cdot & \beta' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \alpha'' & \cdot & \cdot & \cdot & \beta'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \alpha' & \cdot & \cdot & \cdot & \beta' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \alpha'' & \cdot & \cdot & \cdot & \beta'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\beta'' & \cdot & \cdot & \cdot & \alpha'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \beta' & \cdot & \cdot & \cdot & \alpha' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \beta'' & \cdot & \cdot & \cdot & \alpha'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \beta' & \cdot & \cdot & \cdot & \alpha' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma' & \cdot & \cdot & \cdot & \delta' & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma'' & \cdot & \cdot & \cdot & \delta'' & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma' & \cdot & \cdot & \cdot & \delta' & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma'' & \cdot & \cdot & \cdot & \delta'' \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta'' & \cdot & \cdot & \cdot & \gamma'' & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta' & \cdot & \cdot & \cdot & \gamma' & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta'' & \cdot & \cdot & \cdot & \gamma'' & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta' & \cdot & \cdot & \cdot & \gamma'
\end{pmatrix}.$$

For every choice of parameters in $\mathfrak{m}$ the matrix $c_Q \circ (\mathrm{id}_V^{\otimes 2} + g)$ satisfies the Yang-Baxter equation modulo $\mathfrak{m}^2$. Two such deformations are gauge equivalent if and only if they share the same values $\alpha = \alpha' + \alpha''$, $\beta = \beta' + \beta''$, $\gamma = \gamma' + \gamma''$, $\delta = \delta' + \delta''$. They satisfy the Yang-Baxter equation to order $\mathfrak{m}^3$ if and only if $\alpha' = \alpha''$, $\beta' = \beta''$, $\gamma' = \gamma''$, $\delta' = \delta''$.

Example 7.7 (colouring polynomials)**.** We conclude with an example where the modular case provides non-trivial diagonal deformations and interesting knot invariants arise at the infinitesimal level.

Consider the alternating group $G = A_5$ and the conjugacy class $Q = (12345)^G$ of order 12. The knot invariant associated to c_Q counts for each knot $K \subset \mathbb{S}^3$ the number of knot group representations $\pi_1(\mathbb{S}^3 \smallsetminus K) \to G$ sending meridians of K to elements of Q. According to [15] the operator c_Q has only trivial deformations over $\mathbb{Q}[[h]]$ or any ring $\mathbb{A}$ with $5! \in \mathbb{A}^\times$.

The modular case is more interesting: if we consider $\mathbb{A} = \mathbb{Z}_5[h]/(h^2)$, then c_Q does allow non-trivial deformations that are topologically interesting [16, Exm. 1.3]. According to Theorem 1.3, *all* infinitesimal deformations of c_Q are encoded by rack cohomology, which has been intensely studied in recent years and is fairly well understood. The associated knot invariants can be identified as *colouring polynomials*, counting knot group representations $\pi_1(\mathbb{S}^3 \smallsetminus K) \to G$ while keeping track of longitudinal information [16].

8. Open questions

8.1. **From infinitesimal to complete deformations.** As explained in §3, rack cohomology $H^2_{\mathrm{R}}(Q;\mathbb{K})$ encodes infinitesimal deformations of c_Q, that is, deformations over $\mathbb{A} = \mathbb{K}[h]/(h^2)$. Even at the infinitesimal level this approach leads to interesting knot invariants, as illustrated by Example 7.7 above. In the framework of Yang-Baxter deformations, the following generalization appears natural:

Question 8.1. What can be said about complete deformations, that is, deformations of c_Q over the power series ring $\mathbb{K}[\![h]\!]$ or the p-adic integers $\mathbb{Z}_p$?

Higher order obstructions are encoded in $H^3_{\mathrm{YB}}(c_Q, \mathfrak{m}^2/\mathfrak{m}^3)$, and in general seem to be non-trivial. For deformations over $\mathbb{Q}[\![h]\!]$ this question has been completely solved in [15]. The modular case is still open and potentially more interesting.

Question 8.2. Given a deformation of c_Q, what sort of topological information is contained in the associated knot invariant?

For knot invariants coming from rack or quandle cohomology, this question was answered in [16]. For non-diagonal deformations the question is still open. Notice that the problem gets more complicated and more intriguing as we approach the quantum case: the closer Q is to the trivial quandle, the more deformations will appear. Their topological interpretation, however, becomes more difficult, and for the time being remains mysterious.

8.2. **From racks to biracks.** Given a set Q and a bijective map $c\colon Q \times Q \to Q \times Q$, we can formulate the set-theoretic Yang-Baxter equation [12] as

$$(\mathrm{id} \times c)(c \times \mathrm{id})(\mathrm{id} \times c) = (c \times \mathrm{id})(\mathrm{id} \times c)(c \times \mathrm{id}).$$

In general c will have the form $c(x,y) = (x \rhd y, x \lhd y)$ with two binary operations $\rhd, \lhd\colon Q \times Q \to Q$, see [18, 36] for details. Recently, Kauffman's theory of virtual knots [33] has rekindled interest in such set-theoretic solutions $(Q, \rhd, \lhd)$ called *biracks* or *biquandles* [20, 34]. Racks correspond to the case where the operation $x \rhd y = y$ is trivial whereas $x \lhd y = x^y$ is the rack operation.

Question 8.3. Can our results be extended to set-theoretic solutions of the Yang-Baxter equation that do not come from racks?

Our notion of Yang-Baxter cohomology [15] has been conceived for arbitrary Yang-Baxter operators, and in particular it covers set-theoretic solutions such as biracks and biquandles above. The restricted setting of diagonal deformations has been studied by Carter et al. [8]. More general deformations still need to be examined.

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